1.0

4.5
5.0

2.8

2.5

3.2

2.2

3.6

1.1

4.0

2.0

1.8

1.25

1.4

1.6

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LINEARIZING NONLINEAR Ø-1 PROGRAMS.

by

Egon Balas

and
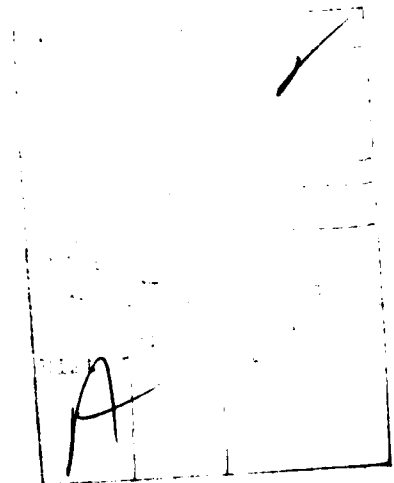
Joseph B. Mazzola

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Abstract

Any real-valued nonlinear function in 0-1 variables can be rewritten as a multilinear function. We discuss classes of lower and upper bounding linear expressions for multilinear functions in 0-1 variables. For any multilinear inequality in 0-1 variables, we define an equivalent family of linear inequalities. This family contains the set of generalized covering inequalities defined by Granot and Hammer. Several results concerning the relative strengths of inequalities within this family are presented. An algorithm for the general multilinear 0-1 program is given, and computational experience with the algorithm applied to randomly generated problems is discussed. The use of the general procedure as an effective heuristic for multilinear 0-1 programs is also demonstrated.

LINEARIZING NONLINEAR 0-1 PROGRAMS

by

Egon Balas and Joseph B. Mazzola

1. Introduction

It is well known [16] that a real-valued function $f(x)$ in 0-1 variables can be rewritten as a multilinear function in the same variables, i.e.,

$$(1) \qquad f(x) = \sum_{j \in N} a_j \left(\prod_{i \in Q_j} x_i \right), \quad x_i = 0 \text{ or } 1, \quad i \in \bigcup_{j \in N} Q_j,$$

where a_j, $j \in N$, are real numbers, and π means product. Thus, without loss of generality, when discussing nonlinear 0-1 programs it is sufficient to consider the general multilinear program

$$\text{maximize} \quad \sum_{j \in N_0} a_{0j} \left(\prod_{i \in Q_{0j}} x_i \right)$$

(MLP)
$$\sum_{j \in N_k} a_{ij} \left(\prod_{i \in Q_{kj}} x_i \right) \leq b_k, \quad k \in K$$

$$x_i = 0 \text{ or } 1, \, i \in Q = \bigcup_{\substack{k \in K \cup \{0\} \\ j \in N_k}} Q_{kj}$$

where all coefficients are integer. Further, without loss of generality we may assume that the objective function in (MLP) is linear, since it can always be linearized by introducing a new (integer) variable z and amending the constraint set K by one new constraint involving z (or its binary expansion) and the nonlinear part of the objective function.

The subject of nonlinear 0-1 programming or 0-1 polynomial programming, as it is sometimes called, has received a fair amount of attention in the literature (see for example [1-4, 10-16, 18, 24]). For a survey of the area, see Hansen [19].

Applications involving nonlinear 0-1 programming arise in various areas. Such formulations have been used in modular design [8], [9], capital budgeting under uncertainty [21], project scheduling [22], cluster analysis [23], diagnostic testing [20], media selection [25], etc. Naturally, quadratic zero-one programming and the quadratic assignment problem are also examples of nonlinear 0-1 programs.

In this paper we present a new linearization for nonlinear functions and inequalities in 0-1 variables, that does not introduce new variables. We then discuss an algorithm based on this approach, and computational experience with it.

We begin by introducing a family of linear (and affine) upper and lower bounding functions for the multilinear function $f(x)$. Several results defining classes of such functions are stated, and then properties of these classes of bounding functions are discussed.

In the next section, we consider the multilinear inequality

$$(2) \qquad f(x) = \sum_{j \in N} a_j (\pi_{i \in Q_j} x_i) \leq b,$$

where a_j, $j \in N$, and b, are integers, and introduce a family of linear inequalities which is equivalent to (2). This family subsumes the generalized covering inequalities for (2) introduced by Granot and Hammer [16]. We then investigate properties relating the strengths of inequalities within this family, with the objective of obtaining a smaller cardinality (or more compact) linearization. We also discuss the use of this linearization on certain classes of multilinear inequalities implied by (2) in order to obtain alternate linearizations of (2). Various examples illustrating the fundamental concepts are presented.

Following this, a general algorithm for solving multilinear 0-1 programs is presented in section three. This algorithm uses the new linearizations introduced in section two, and it also has other new features. Specifically, the effectiveness of the algorithm is greatly enhanced by the incorporation of a heuristic to obtain near optimal solutions to the 0-1 linear relaxations at each iteration. Once a heuristic solution to the current linear relaxation is feasible to (ML), the algorithm switches to an exact solution technique. However, at that point the procedure can be terminated if desired, i.e., the procedure can also be used as a heuristic to find "good" solutions to large multilinear 0-1 programs.

In the final section, we present computational results with three different versions of the algorithm. In particular, a version which uses linear inequalities arising from the new linearization of section two is compared with one which utilizes only generalized covering inequalities, like the procedure of Granot, Granot and Kallberg [15] (see also [14]). The version stemming from the direct application of the new linearization is found to be superior to the generalized covering approach when applied to multilinear 0-1 programs having more than a few terms per constraint, and the difference in performance tends to increase with the number of terms. Thus, our procedure opens up a new class of multilinear 0-1 programs to exact solution. Additionally, we present computational results concerning the use of the procedure as a heuristic. Typically, the heuristic solutions obtained were (guaranteed to be) within 3% of optimality, and for those cases in which the optimal solution was known, the heuristic solution was on the average within 0.25% of the optimal integer solution.

2. Upper and Lower Bounding Affine Functions

Consider the multilinear function

$$(1) \qquad f(x) = \sum_{j \in N} a_j (\prod_{i \in Q_j} x_i),$$

where $x_i = 0$ or 1, $i \in Q_j$, $j \in N$. For any $M \subseteq N$, let $Q_M = \bigcup_{j \in M} Q_j$, and let $Q = Q_N$, $q = |Q|$. Also, for any $x \in \{0,1\}^q$, let $Q(x) = \{i \in Q | x_i = 1\}$. $Q(x)$ is often referred to as the support of x. A function $g(x)$ is said to be an upper (lower) bounding function for $f(x)$ if $f(x) \le g(x)$ $(f(x) \ge g(x))$ for all $x \in \{0,1\}^q$.

In what follows, summation over the empty set is always taken to yield zero.

__Theorem 1.__ Let $f(x)$ be as in (1), with $a_j > 0$, $j \in N$, and for every $M \subseteq N$, define

$$(3) \qquad g_M(x) = \sum_{i \in Q_M} (\overline{\sum_{j \in M | i \in Q_j}} a_j) x_i - \sum_{j \in M} (|Q_j| - 1) a_j.$$

Then every $x \in \{0,1\}^q$ satisfies the inequality

$$(4)_M \qquad f(x) \ge g_M(x)$$

for every $M \subseteq N$, and $(4)_M$ holds with equality if and only if

$$(5) \qquad \{j \in N | Q_j \subseteq Q(x)\} \subseteq M \subseteq \{j \in N \mid |Q_j \setminus Q(x)| \le 1\}.$$

__Proof.__ Let $x \in \{0,1\}^q$ be fixed. For any $M \subseteq N$, let $Q_M(x) = Q_M \cap Q(x)$. Then

$$f(x) = \overline{\sum_{j \in N | Q_j \subseteq Q(x)}} a_j.$$

On the other hand,

$$(6) \qquad g_M(x) = \sum_{j \in M} (1 - |Q_j| + |Q_j \cap Q_M(x)|) a_j \leq \sum_{j \in N | Q_j \subseteq Q(x)} a_j,$$

since $M \subseteq N$, $Q_M(x) \subseteq Q(x)$, and for all $j \in N$, $|Q_j| \geq 1$ and $a_j > 0$. Thus $(4)_M$ holds for all $M \subseteq N$.

Further, for given x assume M satisfies (5), and let

$$M_k = \{j \in M \mid |Q_j \backslash Q(x)| = k\}, \quad k = 0,1.$$

Then both sides of the inequality (6) are equal to

$$\sum_{j \in M_0} a_j$$

hence $(4)_M$ holds with equality.

Conversely, if for the given x, M does not satisfy (5), let

$$M^+ = M \backslash (M_0 \cup M_1) \qquad \text{and} \qquad N^* = \{j \in N \backslash M \mid Q_j \subseteq Q(x)\}.$$

Then (6) becomes

$$\sum_{j \in M_0} a_j + \sum_{j \in M^+} (1 - |Q_j| + |Q_j \cap Q_M(x)|) a_j \leq \sum_{j \in M_0} a_j + \sum_{j \in N^*} a_j$$

or

$$(7) \qquad \sum_{j \in N^*} a_j + \sum_{j \in M^+} k_j a_j \geq 0,$$

where

$$k_j = |Q_j \backslash Q_M(x)| - 1.$$

Now by the definition of M^+, $k_j > 0$ for all $j \in M^+$, and since M violates (5), we have

$$N^* \cup M^+ \neq \emptyset.$$

Thus (7) can be replaced by

$$\sum_{j \in N^*} a_j + \sum_{j \in M^+} k_j a_j > 0,$$

i.e., $(4)_M$ holds as strict inequality.$\|$

Remark 1. $g_M(x) = \sum_{j \in M} g_{\{j\}}(x).$

Proof. By applying the definition of $g_M(x)$ to $\{j\}$ for each $j \in M$.$\|$

Remark 2. For every $x \in \{0,1\}^q$, there exists some $M \subseteq N$ such that $f(x) = g_M(x)$.

Proof. Set $M = \{j \in N | Q_j \subseteq Q(x)\}$. Then x and M satisfy (5), hence $f(x) = g_M(x).\|$

A set $\mathcal{P}$ of lower (upper) bounding functions $p(x)$ for $f(x)$ will be called **complete** if for every $x \in \{0,1\}^q$ there exists $p \in \mathcal{P}$ such that $p(x) = f(x)$. From Remark 2, the set $\mathcal{L} = \{g_M(x) | M \subseteq N\}$ is complete. Since $\mathcal{L}$ is fairly large $(|\mathcal{L}| = 2^{|N|})$, it is of interest to find proper subsets of $\mathcal{L}$ that are complete. Next we identify one such subset.

For any $M \subseteq N$, define

$$E_i(M) = \{j \in N \, | \, |Q_j \setminus Q_M| = i\}, \quad i = 1,\ldots,p,$$

where

$$p = \max_{j \in N} \{|Q_j \setminus Q_M|\}.$$

Clearly, $M \subseteq E_0(M)$ for any $M \subseteq N$. Further, denote

$$E(M) = E_0(M) \cup E_1(M),$$

and note that for arbitrary subsets M_1, $M_2 \subseteq N$, $M_1 \neq M_2$ does **not** imply $E(M_1) \neq E(M_2)$.

Consider now the family

$$\mathcal{L}_0 = \{g_{E(M)}(x) \,|\, M \subseteq N\}$$

of lower bounding functions for $f(x)$, whose cardinality is typically much smaller than that of $\mathcal{L}$.

<u>Theorem 2</u>. Let $f(x)$ be as in (1), with $a_j > 0$, $j \in N$. Then $\mathcal{L}_0$ is a complete set of lower bounding functions for $f(x)$.

<u>Proof</u>. For a given $x \in \{0,1\}^q$, define $L = \{j \in N \,|\, Q_j \subseteq Q(x)\}$. Then $g_{E(L)} \in \mathcal{L}_0$, and setting $M = E(L)$ in (5) shows that $f(x) = g_{E(L)}(x)$. Since this is true for every $x \in \{0,1\}^q$, $\mathcal{L}_0$ is complete.$\|$

<u>Remark 3</u>. For every $M \subseteq N$, there exists some $x \in \{0,1\}^q$ such that $f(x) = g_{E(M)}(x)$.

<u>Proof</u>. For given $M \subseteq N$, let $\hat{x}$ be defined by $Q(\hat{x}) = \underset{j \in M}{\cup} Q_j$. Then

$$\{j \in N \,|\, Q_j \subseteq Q(\hat{x})\} \subseteq E(M) \subseteq \{j \in N \,|\, |Q_j \backslash Q(\hat{x})| \leq 1\},$$

and hence, from Theorem 1, $f(\hat{x}) = g_{E(M)}(\hat{x})$.$\|$

Note that, while every lower bounding function in $\mathcal{L}_0$ is "attained" by $f(x)$ for some $x \in \{0,1\}^q$, the same is <u>not</u> true in general with respect to the larger family $\mathcal{L}$. Thus, let

$$f(x) = x_1 x_2 x_3 + x_4 x_5 + x_1 x_4 + x_1 x_5 + x_2 x_5 + x_3 x_4,$$

and choose $M = \{1,2\}$, where $Q_1 = \{1,2,3\}$, $Q_2 = \{4,5\}$. Then the lower bounding function

$$g_{\{1,2\}}(x) = x_1 + x_2 + x_3 + x_4 + x_5 - 3$$

is not equal to $f(x)$ for any $x \in \{0,1\}^5$.

Next we illustrate the families $\mathcal{L}$ and $\mathcal{L}_0$ on an example.

Example 1. Let

$$f(x) = 3x_1 x_2 x_3 + 2x_1 x_4 + x_2 x_3 x_4.$$

Then

$$g_{\{1,2,3\}} = 5x_1 + 4x_2 + 4x_3 + 3x_4 - 10$$

$$g_{\{1,2\}} = 5x_1 + 3x_2 + 3x_3 + 2x_4 - 8$$

$$g_{\{2,3\}} = 2x_1 + x_2 + x_3 + 3x_4 - 4$$

$$g_{\{1,3\}} = 3x_1 + 4x_2 + 4x_3 + x_4 - 8$$

$$g_{\{1\}} = 3x_1 + 3x_2 + 3x_3 \qquad - 6$$

$$g_{\{2\}} = 2x_1 \qquad\qquad + 2x_4 - 2$$

$$g_{\{3\}} = \qquad x_2 + x_3 + x_4 - 2$$

$$g_{\emptyset} = \qquad\qquad\qquad 0$$

A complete system of lower bounding functions consists of

$$\mathcal{L}_0 = \{g_{\{1,2,3\}}, g_{\{2\}}, g_{\emptyset}\},$$

since for all $M \subset \{1,2,3\}$, $M \neq \{2\}$, $\emptyset$, we have $E(M) = \{1,2,3\}$. ‖

Next we turn to upper bounding linear functions for $f(x)$. Let φ be a mapping that associates to every Q_j, $j \in N$, some $i \in Q_j$, i.e., $\varphi(Q_j) = i(j) \in Q_j$, and let Φ be the set of all such mappings.

Theorem 3. Let $f(x)$ be as in (1), with $a_j > 0$, $j \in N$. For $\varphi \in \Phi$, define

$$h_{\varphi}(x) = \sum_{j \in N} a_j x_{i(j)},$$

where $i(j) = \varphi(Q_j)$. Then every $x \in \{0,1\}^q$ satisfies the inequality

$$(8)_{\varphi} \qquad f(x) \leq h_{\varphi}(x)$$

for every $\varphi \in \Phi$, and $(8)_{\varphi}$ holds as equality if and only if

(9) $\quad\quad i(j) \in Q_j \setminus Q(x)$

for all $j \in N$ such that $Q_j \setminus Q(x) \neq \emptyset$.

     Proof. For a given $x \in \{0,1\}^q$, define $M = \{j \in N \mid Q_j \subseteq Q(x)\}$. Then for every $\varphi \in \Phi$,

$$(10) \quad\quad f(x) = \sum_{j \in M} a_j \quad\quad\quad\quad \text{(by the choice of M)}$$

$$= \sum_{j \in M} a_j x_{i(j)} \quad\quad\quad \text{(where } i(j) = \varphi(Q_j))$$

$$\leq \sum_{j \in N} a_j x_{i(j)} = h_\varphi(x) \quad\quad \text{(since } a_j > 0, \ j \in N),$$

i.e., $(8)_\varphi$ is satisfied.

     If $i(j) \in Q_j \setminus Q(x)$ for all $j \in N$ such that $Q_j \setminus Q(x) \neq \emptyset$, then $x_{i(j)} = 0$, $\forall j \in N \setminus M$, and the inequality in (10), hence in $(8)_\varphi$, holds as equality. Conversely, if $i(j) \in Q_j \cap Q(x)$ for some $j \in N \setminus M$, then $x_{i(j)} = 1$ and $(8)_\varphi$ holds as strict inequality, since $a_j > 0$, $j \in N$. Since this argument applies to every $x \in \{0,1\}^q$, the proof is complete. ‖

     Remark 4. If $f(x)$ is as in (1) but with $a_j < 0$, $j \in N$, then for every $\varphi \in \Phi$, $h_\varphi(x)$ is a lower bounding linear function for $f(x)$.

     Proof. Applying Theorem 3 to $-f(x)$ yields $-f(x) \leq -h_\varphi(x)$, $\forall \varphi \in \Phi$. ‖

     Remark 5. For every $\varphi \in \Phi$ there exists some $x \in \{0,1\}^q$ for which $f(x) = h_\varphi(x)$.

     Proof. Both $x = 0$ and $x = e$, where $e = (1,\ldots,1)$, produce equality in $(8)_\varphi$ for all $\varphi \in \Phi$.

     Remark 6. For every $x \in \{0,1\}^q$, there exists some $\varphi \in \Phi$ such that $f(x) = h_\varphi(x)$.

     Proof. Use any mapping satisfying $\varphi(Q_j) \in Q_j \setminus Q(x)$ for all $j \in N$ such that $Q_j \setminus Q(x) \neq \emptyset$; then $(8)_\varphi$ holds as equality. ‖

Thus the family

$$\mathcal{U} = \{h_\varphi(x) \mid \varphi \in \Phi\}$$

of upper bounding functions for $f(x)$ is complete in the above defined sense.

There is actually a more general class of upper bounding linear functions for $f(x)$, namely

Remark 7. Let λ_{ji}, $i \in Q$, $j \in N$ be nonnegative numbers satisfying

(11)
$$\sum_{i \in Q_j} \lambda_{ji} = 1, \qquad j \in N,$$

and define

$$h(\lambda, x) = \sum_{j \in N} a_j \left(\sum_{i \in Q_j} \lambda_{ji} x_i \right).$$

Then every $x \in \{0,1\}^q$ satisfies the inequality

$$f(x) \leq h(\lambda, x)$$

for every $\lambda \geq 0$ satisfying (11).

Proof. For any x, define $Q(x)$ as before; then

$$f(x) = \overline{\sum_{j \in N \mid Q_j \subseteq Q(x)}} a_j$$

and

$$h(\lambda, x) = \sum_{j \in N} a_j \left(\overline{\sum_{i \in Q_j \cap Q(x)}} \lambda_{ji} \right)$$

$$= \overline{\sum_{j \in N \mid Q_j \subseteq Q(x)}} a_j \left(\sum_{i \in Q_j} \lambda_{ji} \right) + \overline{\sum_{j \in N \mid Q_j \not\subseteq Q(x)}} a_j \left(\overline{\sum_{i \in Q_j \cap Q(x)}} \lambda_{ji} \right)$$

$$\geq \overline{\sum_{j \in N \mid Q_j \subseteq Q(x)}} a_j = f(x). \|$$

The family $\mathcal{U}$ introduced earlier consists of those $h(\lambda, x)$ such that

$$\lambda_{ji} = \begin{cases} 1 & \text{for } i = i(j) \\ 0 & \text{for } i \neq i(j). \end{cases}$$

Since the set $\mathcal{U}$ is complete and already very large (not excluding repetitions, $|\mathcal{U}| = \underset{j \in N}{\pi} |Q_j|$), we will not consider further the more general class of functions defined in Remark 7, but rather move in the opposite direction of identifying a proper subset of $\mathcal{U}$ that is complete.

For any $\varphi \in \Phi$, define

$$I(\varphi) = \{i \in Q \mid i = \varphi(Q_j) \text{ for some } j \in N\}.$$

We will say that a mapping $\varphi \in \Phi$ is _sequence-related_ if there exists a permutation $< i_1, \ldots, i_m >$ of the elements of $I(\varphi)$, with the property that for $k = 1, \ldots, m$, $i_k = \varphi(Q_j)$ for __all__ $j \in N$ such that $i_k \in Q_j$ but $i_\ell \notin Q_j$ for $\ell = 1, \ldots, k-1$. We will say that $s_\varphi = < i_1, \ldots, i_m >$ is _the sequence associated with_ the mapping φ. To put it differently, a mapping $\varphi \in \Phi$ is sequence-related if it can be generated as follows: choose some $i \in Q$ and set $i = \varphi(Q_j)$ for all $j \in N$ such that $i \in Q_j$; then remove from Q all subsets Q_j containing i, and apply the procedure to the redefined set Q, stopping when Q becomes empty.

Example 2. Let $Q_1 = \{1,2,3\}$, $Q_2 = \{1,4,5\}$, $Q_3 = \{2,5\}$. Then each of the mappings

$$\varphi_1(Q_1) = 1, \quad \varphi_1(Q_2) = 1, \quad \varphi_1(Q_3) = 2$$

$$\varphi_2(Q_1) = 1, \quad \varphi_2(Q_2) = 5, \quad \varphi_2(Q_3) = 5$$

$$\varphi_3(Q_1) = 1, \quad \varphi_3(Q_2) = 4, \quad \varphi_3(Q_3) = 2$$

is sequence-related, with the associated sequences $\{1,2\}$, $\{5,1\}$ and $\{4,1,2\}$ respectively; but the mapping

$$\varphi_4(Q_1) = 1, \quad \varphi_4(Q_2) = 5, \quad \varphi_4(Q_3) = 2$$

is not sequence-related, since for any permutation of the indices $1,5,2$, the first index does not represent all sets Q_j in which it is contained (1 is contained in Q_1 and Q_2, 5 in Q_2 and Q_3, 2 in Q_1 and Q_3).

Let $\Phi' = \{\varphi \in \Phi \mid \varphi$ is sequence-related$\}$, and

$$\mathcal{U}' = \{h_\varphi(x) \mid \varphi \in \Phi'\}.$$

It can be shown (see below) that $\mathcal{U}'$ is a complete set of upper bounding functions for $f(x)$. However, it turns out that Φ' can be further restricted without losing completeness.

Let V be an arbitrary set with $|V| = v$, and let θ be the set of all permutations of the elements of V. For any $S \subseteq V$, a permutation $p \in \theta$ will be said to <u>represent</u> in θ the 2-partition $(S, V\backslash S)$ of V, if every element of S precedes every element of $V\backslash S$ in p. In other words, the permutation $p = \langle i_1, \ldots, i_v \rangle$ represents the partition $(S, V\backslash S)$ if $i_k \in S$ and $i_\ell \in V\backslash S$ imply $k < \ell$. A set of permutations $P \subseteq \theta$ will be called <u>representative</u> (of the 2-partitions of V), if for every $S \subseteq V$, the partition $(S, V\backslash S)$ is represented in P.

<u>Example 3</u>. Representative sets of permutations for $V_1 = \{1,2,3\}$ and $V_2 = \{1,2,3,4\}$ are P_1 and P_2,

P_1:	1 2 *	P_2:	1 2 3 *
	1 3 *		1 2 4 *
	<u>2 3 *</u>		1 3 4 *
	3 * *		<u>2 3 4 *</u>
			1 4 * *
			2 4 * *
			<u>3 4 * *</u>
			4 * * *

where a star in some $p \in P_i$ stands for an arbitrary element of V_i not yet used in p.

While the cardinality of $\mathcal{P}$ is $v!$, that of a representative subset $P \subseteq \mathcal{P}$ is only 2^{v-1}.

Consider now the set of sequence-related mappings Φ'. For any $S \subseteq Q$, we say that a mapping $\varphi \in \Phi'$, with associated sequence $s_\varphi = <i_1,\ldots,i_m>$, represents the 2-partition $(S, Q \backslash S)$, if s_φ is a subsequence of some permutation $p = <j_1,\ldots,j_q>$ of the elements of Q, that represents $(S, Q \backslash S)$. A set $\Psi \subset \Phi'$ of sequence-related mappings will be called <u>representative</u> (of the 2-partitions of Q) if every 2-partition of Q is represented in Ψ.

Now let $\Psi \subset \Phi'$ be representative, and define

$$\mathcal{U}_0 = \{h_\varphi(x) \mid \varphi \in \Psi\}.$$

<u>Theorem 4</u>. Let $f(x)$ be as in (1), with $a_j > 0$, $j \in N$. Then $\mathcal{U}_0$ is a complete set of upper bounding functions for $f(x)$.

<u>Proof</u>. For an arbitrary $x \in \{0,1\}^q$ let $\varphi \in \Psi$ be the mapping that represents the partition $(Q \backslash Q(x), Q(x))$ of Q (here, as before, $Q(x) = \{j \in Q \mid x_j = 1\}$), let $s_\varphi = <i_1,\ldots,i_m>$ be the sequence associated with φ, and let $p = <j_1,\ldots,j_q>$ be a permutation of the elements of Q that represents $(Q \backslash Q(x), Q(x))$, such that s_φ is a subsequence of p.

Now if $i_m \in Q \backslash Q(x)$, then $i_\ell \in Q \backslash Q(x)$ for $\ell = 1,\ldots,m$, and $h_\varphi(x) = 0 = f(x)$. Otherwise, let h and k, respectively, be the greatest integers such that $i_h \in Q \backslash Q(x)$ and $j_k \in Q \backslash Q(x)$. Then $Q \backslash Q(x) = \{j_1,\ldots,j_k\}$, and $\{i_1,\ldots,i_h\} \subseteq Q \backslash Q(x)$, $\{i_{h+1},\ldots,i_m\} \subseteq Q(x)$. Denote

$$N_0 = \left\{ j \in N \mid \varphi(Q_j) \in \{i_1,\ldots,i_h\} \right\},$$

$$N_1 = \left\{ j \in N \mid \varphi(Q_j) \in \{i_{h+1},\ldots,i_m\} \right\},$$

Clearly, $N_0 \cup N_1 = N$. Since φ is related to the sequence s_φ, $i_\ell \notin Q_j$ for $\ell \in \{1,\ldots,h\}$ and $j \in N_1$, hence $Q_j \subseteq Q(x)$ for all $j \in N_1$. Therefore $Q_j \setminus Q(x) = \emptyset$ implies $j \in N_0$, which in turn implies $\varphi(Q_j) \in Q \setminus Q(x)$; i.e., condition (9) of Theorem 3 holds for all $j \in N$ such that $Q_j \setminus Q(x) \neq \emptyset$. Therefore $f(x) = h_\varphi(x)$. ‖

Example 4. As in example 1, let

$$f(x) = 3x_1 x_2 x_3 + 2x_1 x_4 + x_2 x_3 x_4.$$

The set Φ of all mappings that associate to each of the sets Q_j, $j = 1,2,3$, one of its elements, contains $|Q_1| \times |Q_2| \times |Q_3| = 18$ elements, and the corresponding 18 upper bounding functions $h_\varphi(x)$, $\varphi \in \Phi$, happen to be pairwise distinct. However, a complete set $\mathcal{U}_0$ of upper bounding functions is defined by the representative set of sequence-related mappings associated with the set P_2 of Example 2 (where $Q = \{1,2,3,4\}$ plays the role of V_2):

$$
\begin{aligned}
h_{\varphi_1}(x) &= 5x_1 + x_2 \\
h_{\varphi_2}(x) &= 5x_1 \quad\quad + x_3 \\
h_{\varphi_3}(x) &= 5x_1 \quad\quad\quad\quad + x_4 \\
h_{\varphi_4}(x) &= \quad\quad 4x_2 \quad\quad + 2x_4 \\
h_{\varphi_5}(x) &= \quad\quad\quad\quad 4x_3 + 2x_4 \\
h_{\varphi_6}(x) &= \quad\quad\quad\quad 3x_3 + 3x_4.
\end{aligned}
$$

The mappings φ_k, $k = 1,\ldots,6$, correspond to the following permutations $p \in P_2$:

k	1	2	3	4	5	6
$p \in P$	1 2 3 *	1 3 4 *	1 4 * *	2 3 4 *	3 4 * *	4 * * *
	1 2 4 *			2 4 * *		

Where a mapping φ_k corresponds to more than one $p \in P_2$ as for $k = 1,4$, this is because different permutations containing a certain subsequence $< i_1,\ldots,i_m >$ give rise to a single mapping $\varphi \in \Psi$ related to that subsequence. Thus, in the case of $k = 1$, $s_\varphi = < 1,2 >$, and in the case of $k = 4$, $s_\varphi = < 2,4 >$.

At this point we mention that there is another class of upper bounding (affine) functions for $f(x)$, that one can derive by using the following observation.

Let Q be a set whose elements are ordered in some arbitrary way, $Q = \{1,\ldots,q\}$, and let a be an arbitrary positive scalar. Then, denoting $\bar{x}_i = 1 - x_i$ for $i \in Q$, one can write

$$(12) \qquad -a \prod_{i \in Q} x_i = a(\bar{x}_q \prod_{i=1}^{q-1} x_i + \bar{x}_{q-1} \prod_{i=1}^{q-2} x_i + \ldots + \bar{x}_2 x_1 + \bar{x}_1 - 1).$$

Note that the right hand side of (12) has $q = |Q|$ variable terms (each one containing exactly one complemented variable) and a constant term.

Thus for any $f(x)$ of the form (1) with $a_j > 0$, $j \in N$, using (12) one can write

$$-f(x) = p_\alpha(x,\bar{x}),$$

where p_α is a multilinear function of the variables x_i and their complements $\bar{x}_i$, $i \in Q_j$, $j \in N$, with coefficients $a_j > 0$ and with $|Q_j|$ terms for each $j \in N$. The subscript α refers to the particular ordering of the sets Q_j, $j \in N$, which was used in (12) to derive $p_\alpha(x,\bar{x})$. Taking any α and applying Theorem 1, one can derive a family of lower bounding functions $g_M(x,\bar{x})$ for $p_\alpha(x,\bar{x})$, one for each set of the form $M = \bigcup_{j \in N} M_j$, where $M_j \subseteq Q_j$, $j \in N$. Substituting for $\bar{x}_i$, $i \in Q_j$, $j \in N$, one obtains the corresponding functions $\hat{g}_M(x)$ ($= g_M(x,\bar{x})$) in the variables x_i, and by changing signs, the affine upper bounding functions $-\hat{g}_M(x)$ for $f(x)$.

For a given $f(x)$ of the form (1), there are $\pi\limits_{j\in N}(|Q_j|!)$ different functions $p_\alpha(x,\bar{x})$ such that $p_\alpha(x,\bar{x}) = -f(x)$, and for each α, there are $\pi\limits_{j\in N} 2^{|Q_j|}$ lower bounding functions $g_M(x,\bar{x})$ (not necessarily all distinct) for $p_\alpha(x,\bar{x})$, hence upper bounding functions $-\hat{g}_M(x)$ for $f(x)$. However, as stated in the next theorem, every such upper bounding affine function is dominated by some linear function in the class $\mathcal{U}$.

To simplify the notation, we will assume that $f(x)$ has a single term, i.e., the equation $-f(x) = p_\alpha(x,\bar{x})$ is as in (12). In view of Remark 1, this implies no loss of generality. Further, we shall let each of the $|Q| = q$ terms of $p_\alpha(x,\bar{x})$ be indexed by the index of its (unique) complemented variable.

Theorem 5. Let $M \subseteq Q$, $M = \{i_1,\ldots,i_m\}$, with $i_k < i_\ell$ whenever $k < \ell$. Then

$$ax_{i_m} \leq -\hat{g}_M(x)$$

for every $x \in \{0,1\}^q$.

Proof. Applying Theorem 1 to $p_\alpha(x,\bar{x})$, we obtain the lower bounding function

$$g_M(x,\bar{x}) = a[\sum_{k=1}^{m} (\bar{x}_{i_k} + \sum_{i=1}^{i_k-1} x_i - (i_k - 1)) - 1]$$

or

$$\hat{g}_M(x) = -a[\sum_{k=1}^{m} (x_{i_k} - \sum_{i=1}^{i_k-1} x_i + i_k - 2) + 1].$$

Therefore

$$-\hat{g}_M(x) = ax_{i_m} + a\sigma(x)$$

where

$$(13) \qquad \sigma(x) = (i_m - 1 - \sum_{i=1}^{i_m-1} x_i) + \sum_{k=1}^{m-1} (i_k - 2 + x_{i_k} - \sum_{i=1}^{i_k-1} x_i),$$

and to prove the theorem we have to show that $\sigma(x) \geq 0$ for all $x \in \{0,1\}^q$. Note that for $k = 1,\ldots,m - 1$,

$$i_k - 2 + x_{i_k} - \sum_{i=1}^{i_k-1} x_i \begin{cases} = -1 \text{ if } x_i = 1, \, i = 1,\ldots,i_k - 1, \text{ and } x_{i_k} = 0 \\ \geq 0 \text{ otherwise} \end{cases}$$

and

$$i_m - 1 - \sum_{i=1}^{i_m-1} x_i \begin{cases} = 0 \text{ if } x_i = 1, \, i = 1,\ldots,i_m - 1 \\ \geq 1 \text{ otherwise} \end{cases}$$

Since $x_{i_k} = 0$ for some $k \in \{1,\ldots,i_k - 1\}$ excludes $x_i = 1$, $i = 1,\ldots,i_\ell - 1$ for any $\ell > k$, at most one term under the summation sign in (13) can be negative, and if there exists such a term, then

$$i_m - 1 - \sum_{i=1}^{i_m-1} x_i \geq 1.$$

Thus for any $x \in \{0,1\}^q$, $\sigma(x) \geq 0$, hence $-\hat{g}_M(x) \geq ax_{i_m}$. ‖

The relation (12) can be used in the reverse direction too; i.e., in order to derive a set of lower bounding functions for some $f(x)$ as in (1), with $a_j > 0$, $j \in N$, one can use Theorem 3 to derive a set of upper bounding linear functions $h_\varphi(x,\bar{x})$ for $p_\alpha(x,\bar{x}) = -f(x)$, and then substitute for $\bar{x}_i$, $\forall$ i, to obtain a set of functions $\delta_\varphi(x)$, whose negatives, $-\delta_\varphi(x)$, are lower bounding functions for $f(x)$. In this case one recovers the lower bounding function $g_N(x)$, by using the mapping φ which associates to the index set of each term of $p_\alpha(x,\bar{x})$, the index of its complemented variable. The functions $g_M(x)$, $M \subsetneq N$, can be recovered by using the same mapping for $j \in M$, while for $j \in N \backslash M$ one uses a mapping that produces a lower bounding function identically equal to zero. When the number of terms of $p_\alpha(x,\bar{x})$ corresponding to the jth

term of $f(x)$ is even, this is accomplished by any mapping that produces pairwise complementary images. When it is odd, one has to use the construction of Remark 7 to find a lower bounding function that vanishes for all $x \in \{0,1\}^q$.

All other lower bounding functions that one obtains via this procedure are uninteresting, because they cannot take on a positive value for any $x \in \{0,1\}^q$.

We conclude this section by combining the above results to derive a family of lower bounding functions for $f(x)$ as in (1), with coefficients a_j of arbitrary sign. Let

$$N^+ = \{j \in N \mid a_j > 0\} \qquad , \qquad N^- = \{j \in N \mid a_j < 0\},$$

and

$$f^+(x) = \sum_{j \in N^+} a_j \prod_{i \in Q_j} x_i \qquad , \qquad f^-(x) = \sum_{j \in N^-} a_j \prod x_i .$$

For every $M \subseteq N^+$, let

$$g_M(x) = \sum_{i \in Q_M} (\overline{\sum_{j \in M \mid i \in Q_j}} a_j) x_i - \sum_{j \in M} (|Q_j| - 1) a_j,$$

as in Theorem 1. Let $\bar{\Phi}^-$ be the family of mappings φ that associate to every Q_j, $j \in N^-$, some $i \in Q_j$, and for every $\varphi \in \bar{\Phi}^-$, let

$$h_\varphi^-(x) = \sum_{j \in N^-} a_j x_{i(j)}$$

where $i(j) = \varphi(Q_j)$, $j \in N^-$. The function $h_\varphi^-(x)$ differs from the function $h_\varphi(x)$ of Theorem 3 only in that here the coefficients a_j are negative.

<u>Theorem 6</u>. Let $f(x)$ be as in (1). Then every $x \in \{0,1\}^q$ satisfies

$$(14)_{M,\varphi} \qquad f(x) \geq g_M(x) + h_\varphi^-(x)$$

for every $M \subseteq N^+$ and every $\varphi \in \vartheta^-$. Further, $(14)_{M,\varphi}$ holds as equality if and only if

(15) $\qquad \{j \in N^+ | Q_j \subseteq Q(x)\} \subseteq M \subseteq \{j \in N^+ | |Q_j \backslash Q(x)| \leq 1\}$

and for all $j \in N^-$ such that $Q_j \backslash Q(x) \neq \emptyset$,

(16) $\qquad i(j) \in Q_j \backslash Q(x)$

$\underline{Proof}$. Since $N^+ \cup N^- = N$, we have

$$f(x) = f^+(x) + f^-(x);$$

From Theorem 1, for every $M \subseteq N^+$

(17) $\qquad f^+(x) \geq g_M(x);$

and from Theorem 3, for every $\varphi \in \vartheta^-$

(18) $\qquad f^-(x) \geq h_\varphi^-(x).$

Then adding (17) and (18) yields (14).

Now (17) holds as equality if and only if (15) is satisfied (Theorem 1), while (18) holds as equality if and only if (16) is satisfied for all $j \in N^-$ such that $Q_j \backslash Q(x) \neq \emptyset$. (Theorem 3). But a vector $x \in \{0,1\}^q$ that satisfies both (17) and (18), satisfies $(14)_{M,\varphi}$ with equality if and only if it satisfies with equality both (17) and (18). $\|$

Next we define a subfamily of the lower bounding functions introduced in Theorem 6, that is complete.

For every $M \subseteq N^+$, let $E(M)$ be defined as in Theorem 2, and let

$$\mathcal{L}_0^+ = \{g_{E(M)}(x) | M \subseteq N^+\}.$$

Further, let $\Psi^- \subseteq \Phi^-$ be a representative set of sequence-related mappings as defined earlier, and let

$$\mathcal{U}_0^- = \{h_\varphi^-(x) \mid \varphi \in \Psi^-\}.$$

Finally, define the set

$$\mathcal{B}_0 = \{g_{E(M)}(x) + h_\varphi^-(x) \mid M \subseteq N^+ \text{ and } \varphi \in \Psi^-\}.$$

<u>Theorem 7</u>. $\mathcal{B}_0$ is a complete set of lower bounding functions for $f(x)$.

<u>Proof</u>. Since $\mathcal{L}_0^+$ and $\mathcal{U}_0^-$ are complete, for every $x \in \{0,1\}^q$ there exists some $g_{E(M)}(x) \in \mathcal{L}_0^+$ such that $f^+(x) = g_{E(M)}(x)$, and some $h_\varphi^-(x) \in \mathcal{U}_0^-$ such that $f^-(x) = h_\varphi^-(x)$. But then $g_{E(M)}(x) + h_\varphi^-(x) \in \mathcal{B}_0$ and $f(x) = g_{E(M)}(x) + h_\varphi^-(x)$, hence $\mathcal{B}^0$ is complete. ‖

<u>Example 5</u>. Consider the function

$$f(x) = 8x_1x_2x_3 - 2x_1x_2 + 5x_3x_5x_6 - x_2x_5 + 4x_2\dot{x}_4.$$

The set $\mathcal{L}_0^+$ of lower bounding functions for $f^+(x)$ is

$$8x_1 + 12x_2 + 13x_3 + 4x_4 + 5x_5 + 5x_6 - 30,$$
$$8x_1 + 12x_2 + 8x_3 + 4x_4 - 20,$$
$$5x_3 + 5x_5 + 5x_6 - 10,$$
$$4x_2 + 4x_4 - 4,$$
$$0;$$

while the set $\mathcal{U}_0^-$ of lower bounding functions for $f^-(x)$ is

$$- 2x_1 - x_2,$$
$$- 2x_1 - x_5,$$
$$- 3x_2.$$

Hence a complete set $\mathcal{B}_0$ of lower bounding functions for $f(x)$ consists of the 15 functions obtained by adding any of the 5 functions in $\mathcal{L}_0^+$ to any of the 3 functions in $\mathcal{U}_0^-$. ‖

3. Linearizing Multilinear Inequalities in 0-1 Variables

In this section we linearize the multilinear inequality

$$(2) \qquad f(x) = \sum_{j \in N} a_j \left(\prod_{i \in Q_j} x_i \right) \leq b$$

by defining a family $\mathcal{F}$ of linear inequalities, equivalent to (2) in the sense that a 0-1 vector x satisfies $f(x) \leq b$ if and only if it satisfies the linear inequalities in $\mathcal{F}$. This family is shown to contain the family of generalized covering inequalities defined by Granot and Hammer [16]. We then present several results which relate the strengths of inequalities in $\mathcal{F}$, and which are useful for obtaining a smaller family of linear inequalities equivalent to (2). We first give some definitions.

An inequality A is said to <u>dominate</u> an inequality B if every nonnegative x satisfying A also satisfies B. Further, inequality A <u>strictly</u> dominates inequality B if in addition to A dominating B, there exists some point $\tilde{x}$ such that $\tilde{x}$ satisfies B but not A. We shall also find it useful to define the following weaker notion of dominance.

An inequality A is said to <u>c-dominate</u> an inequality B if every 0-1 point x satisfying A also satisfies B. Further, if A c-dominates B and there exists some 0-1 point satisfying B but not A, then A is said to <u>strictly</u> c-dominate B. It is easily verified that an inequality A can c-dominate an inequality B without A dominating B.

We continue to use the notation introduced earlier. In particular, N^+, N^-, $f^+(x)$ and $f^-(x)$ are as in Theorem 6.

A set $M \subseteq N$ is said to be a <u>cover</u> for the inequality (2), if

$$\sum_{j \in M} |a_j| > b - \sum_{j \in N^-} a_j.$$

A cover M is said to be __minimal__, if T is not a cover for any $T \subsetneq M$.

It follows from this definition that a set $M \subseteq N^+$ is a cover for the inequality

$$(2^+) \qquad f^+(x) \leq b$$

if

$$\sum_{j \in M} a_j > b.$$

__Theorem 8.__ Let

$$\mathcal{C} = \{M \subseteq N^+ | M \text{ is a cover for } (2^+)\},$$

and let $g_M(x)$, $h_\varphi^-(x)$ and $\bar{\Phi}^-$ be as in Theorem 6. Then $x \in \{0,1\}^q$ satisfies (2) if and only if it satisfies

$$(19)_{M,\varphi} \qquad g_M(x) + h_\varphi^-(x) \leq b$$

for every $M \in \mathcal{C}$ and $\varphi \in \bar{\Phi}^-$.

> __Proof.__ From Theorem 6, if $x \in \{0,1\}^q$ satisfies (2), then
>
> $$g_M(x) + h_\varphi^-(x) \leq f(x) \leq b$$

for every $\varphi \in \bar{\Phi}^-$ and every $M \subseteq N^+$, hence every $M \in \mathcal{C}$. This proves the "only if" part of the Theorem. To prove the "if" part, suppose $f(\hat{x}) > b$ for some $\hat{x} \in \{0,1\}^q$. From Theorem 7, there exists $M_0 \subseteq N^+$ and $\varphi_0 \in \bar{\Phi}^-$ such that

$$(20) \qquad g_{M_0}(\hat{x}) + h_{\varphi_0}^-(\hat{x}) = f(\hat{x}) > b,$$

i.e., $\hat{x}$ violates the inequality $(19)_{M_0,\varphi_0}$. It remains to be shown that M_0 is a cover for (2^+). We have

$$\sum_{j \in M_0} |a_j| \geq g_{M_0}(\hat{x}) \qquad \text{(from the definition of } g_{M_0})$$

$$> b - h^-_{\varphi_0}(\hat{x}) \qquad \text{(from (20))}$$

$$\geq b \qquad \text{(since } - h^-_{\varphi_0}(\hat{x}) \geq 0),$$

hence M_0 is a cover for (2^+). ‖

Let $\mathcal{S}$ be the system of linear inequalities $(19)_{M, \varphi}$, for all $M \in \mathcal{C}$ and $\varphi \in \bar{\Phi}^-$. According to Theorem 8, the system $\mathcal{S}$ is equivalent to (has the same solution set as) the nonlinear inequality (2). As one may suspect from Theorem 7, $\mathcal{S}$ is not a minimal set with this property. Indeed, for $M \subseteq N^+$, let $E(M)$ be defined as in Theorem 2; and let $\Psi^- \subseteq \bar{\Phi}^-$ be a set of representative sequence-related mappings, as in Theorem 4. We then have

Theorem 9. Theorem 8 remains true if the system $(19)_{M, \varphi}$, $\varphi \in \bar{\Phi}^-$ and $M \in \mathcal{C}$ is replaced by

$$(21)_{M, \varphi} \qquad g_{E(M)}(x) - h^-_{\varphi}(x) \leq b,$$

for every $M \in \mathcal{C}$ and $\varphi \in \Psi^-$.

Proof. Along the same lines as the proof of Theorem 8, using the fact that, from Theorem 7, there exists $M_0 \subseteq N^+$ and $\varphi_0 \in \Psi^-$ such that

$$g_{E(M_0)}(\hat{x}) + h^-_{\varphi_0}(\hat{x}) = f(\hat{x}). \parallel$$

Since Ψ^- is a proper subset of $\bar{\Phi}^-$, and different sets $M \subseteq N^+$ under certain conditions give rise to the same set $E(M)$, the system $\mathcal{S}_0$ of linear inequalities $(21)_{M, \varphi}$, $M \in \mathcal{C}$, $\varphi \in \Psi^-$, is a proper subset of $\mathcal{S}$, and usually of much smaller cardinality.

<u>Example 6</u>. Consider the inequality (2) with $f(x)$ as in Example 5 and $b = 8$, i.e.,

$$8x_1x_2x_3 + 4x_2x_4 + 5x_3x_5x_6 - 2x_1x_2 - x_2x_5 \leq 8.$$

Denoting $Q_1 = \{1,2,3\}$, $Q_2 = \{2,4\}$, $Q_3 = \{3,5,6\}$, $Q_4 = \{1,2\}$, $Q_5 = \{2,5\}$, the subsets of $N^+ = \{1,2,3\}$ that are covers for (2^+) are $M_1 = \{1,2\}$, $M_2 = \{1,3\}$, $M_3 = \{2,3\}$, $M_4 = \{1,2,3\}$. They correspond to the functions

$$g_{M_1}(x) = 8x_1 + 12x_2 + 8x_3 + 4x_4 - 20$$
$$g_{M_2}(x) = 8x_1 + 8x_2 + 13x_3 + 5x_5 + 5x_6 - 26$$
$$g_{M_3}(x) = 4x_2 + 5x_3 + 4x_4 + 5x_5 + 5x_6 - 14$$
$$g_{M_4}(x) = 8x_1 + 12x_2 + 13x_3 + 4x_4 + 5x_5 + 5x_6 - 30.$$

Further, the set $\bar{\Phi}^-$ consists of the mappings φ_1, φ_2, φ_3, φ_4, giving rise to the 4 functions

$$h_{\varphi_1}^-(x) = -2x_1 - x_2$$

$$h_{\varphi_2}^-(x) = -2x_1 - x_5$$

$$h_{\varphi_3}^-(x) = -3x_2$$

$$h_{\varphi_4}^-(x) = -2x_2 - x_5.$$

Thus the system S equivalent to (2) consists of the 16 linear inequalities

$$g_{M_i}(x) + h_{\varphi_j}^-(x) \leq 8, \quad i = 1,2,3,4; \quad j = 1,2,3,4$$

However, since $\Psi^- = \{\varphi_1, \varphi_2, \varphi_3\}$ and $E(M_1) = \{1,2\}(= M_1)$, $E(M_2) = E(M_3) = E(M_4) = \{1,2,3\}(= M_4)$, the smaller system S_0 equivalent to (2) consists of the 6 inequalities

$$6x_1 + 11x_2 + 8x_3 + 4x_4 \leq 28$$

$$6x_1 + 12x_2 + 8x_3 + 4x_4 - x_5 \leq 28$$

$$8x_1 + 9x_2 + 8x_3 + 4x_4 \leq 28$$

$$6x_1 + 11x_2 + 13x_3 + 4x_4 + 5x_5 + 5x_6 \leq 38$$

$$6x_1 + 12x_2 + 13x_3 + 4x_4 + 4x_5 + 5x_6 \leq 38$$

$$8x_1 + 9x_2 + 13x_3 + 4x_4 + 5x_5 + 5x_6 \leq 38. \Vert$$

It is sometimes useful to consider instead of the nonlinear inequality (2), an equivalent system of (nonlinear) inequalities whose coefficients are all positive. This is easily accomplished by replacing $f^-(x)$ in (2) with the family of lower bounding functions $h_\varphi^-(x)$, $\varphi \in \Psi^-$, and then complementing x_j, $j \in N^-$.

Theorem 10. A vector $x \in \{0,1\}^q$ satisfies (2) if and only if it satisfies

$$(22)_\varphi \qquad f^+(x) + \sum_{j \in N^-} |a_j| \bar{x}_{i(j)} \leq b - \sum_{j \in N^-} a_j,$$

where $i(j) = \varphi(Q_j)$, for every $\varphi \in \Psi^-$.

Proof. The "only if" part follows from the fact that

$$(23) \qquad \sum_{j \in N^-} a_j x_{i(j)} \leq f^-(x)$$

for all $i(j) = \varphi(Q_j)$, $\varphi \in \Psi^-$. The "if" part follows from the fact that the set of lower bounding functions $h_\varphi^-(x)$, $\varphi \in \Psi^-$, for $f^-(x)$, is complete; hence if for some $\hat{x} \in \{0,1\}^q$

$$f(\hat{x}) = f^+(\hat{x}) + f^-(\hat{x}) > b,$$

then there exists $\varphi_0 \in \Psi^-$ such that, putting $i(j) = \varphi_0(Q_j)$,

$$\sum_{j \in N^-} a_j \hat{x}_{i(j)} = f^-(\hat{x}) > b - f^+(\hat{x}),$$

i.e., $\hat{x}$ violates $(22)_{\varphi_0}$. $\Vert$

Remark 8. Theorem 10 remains true if Ψ^- is replaced by $\hat{\Psi}^-$.

Proof. Since $\Psi^- \subset \hat{\Psi}^-$, the "if" part is obviously true; the "only if" part follows from the fact that (23) holds for $\varphi \in \hat{\Psi}^-$. ‖

Note that if we first replace (2) by the family $(22)_\varphi$, $\varphi \in \hat{\Psi}^-$ and then generate the sets $\mathcal{S}_\varphi$ of linear inequalities equivalent to each inequality $(22)_\varphi$, we end up with a set $\mathcal{S}_+ = \bigcup_{\varphi \in \hat{\Psi}^-} \mathcal{S}_\varphi$ of linear inequalities that is a proper superset of the one obtained when we generate the set $\mathcal{S}$ directly from (2). The reason for this is that, applying Theorem 8 to an inequality $(22)_\varphi$, we will generate a linear inequality for every cover $M \subseteq N$ for $(22)_\varphi$, hence for (2); whereas applying it to the inequality (2), we generate linear inequalities only for covers $M \subseteq N^+$ for (2^+). It is easy to see that if $M_1 \subseteq N$ and $M_2 \subseteq N$ are covers for (2), we may have $M_1 \neq M_2$, but $M_1 \cap N^+ = M_2 \cap N^+$. On the other hand, $M \subseteq N^+$ is a cover for (2) if and only if $M \cup N^-$ is a cover for $(22)_\varphi$, $\forall \varphi \in \hat{\Psi}^-$. Thus the system of linear inequalities $\mathcal{S}$, obtained by applying Theorem 8 directly to (2), is that subsystem of $\mathcal{S}_+$ whose inequalities correspond to those covers $M \subseteq N$ such that $N^- \subseteq M$.

Next we turn to another way of using complements of the variables. If we restate the system (19) by complementing the variables x_i, $i \in Q_M$, this operation can be combined with a (trivial) strengthening of some inequalities. For any $\varphi \in \hat{\Psi}^-$, we will denote

$$Q_\varphi = \{ i \in Q \mid i = i(j) \text{ for some } j \in N^- \},$$

where, as before, $i(j) = \varphi(Q_j)$.

Theorem 11. The vector $x \in \{0,1\}^q$ satisfies (2) if and only if it satisfies

$$(24)_{M,\varphi} \qquad \sum_{i \in Q_M} \alpha_i^M \bar{x}_i + \sum_{i \in Q_\varphi} \beta_i x_i \geq \alpha_0^M$$

for every $M \in \mathcal{C}$, $\varphi \in \hat{\Psi}^-$, where

$$\alpha_0^M = \sum_{j \in M} a_j - b,$$

$$\alpha_i^M = \min\{\alpha_0^M, \overline{\overline{\sum_{j \in M | i \in Q_j}}} a_j\} \quad , \qquad i \in Q_M,$$

and

$$\beta_i = \min\{\alpha_0^M, \overline{\overline{\sum_{j \in N^- | i = i(j)}}} |a_j|\}, \qquad i \in Q_\varphi .$$

Proof. Substituting for $g_M(x)$ and $h_\varphi^-(x)$ in $(19)_{M,\varphi}$ their expressions in terms of the coefficients a_j, $j \in N$, yields

$$(19')_{M,\varphi} \qquad \sum_{i \in Q_M} (\overline{\overline{\sum_{j \in M | i \in Q_j}}} a_j) x_i + \sum_{j \in N^-} a_j x_{i(j)} \le b + \sum_{j \in M} (|Q_j| - 1) a_j,$$

where $i(j) = \varphi(Q_j)$, $j \in N^-$, as before. Substituting $1 - \bar{x}_i$ for x_i, $i \in Q_M$, and $- |a_j|$ for a_j, $j \in N^-$, and collecting terms for each $i \in Q_\varphi$, then changing the sign of the inequality, produces

$$(25) \qquad \sum_{i \in Q_M} (\overline{\overline{\sum_{j \in M | i \in Q_j}}} a_j) \bar{x}_i + \sum_{i \in Q_\varphi} (\overline{\overline{\sum_{j \in N^- | i = i(j)}}} |a_j|) x_i \ge \sum_{j \in M} a_j - b,$$

since

$$\sum_{i \in Q_M} (\overline{\overline{\sum_{j \in M | i \in Q_j}}} a_j) 1 = \sum_{j \in M} |Q_j| a_j .$$

Finally, since all coefficients of (25) are positive, each coefficient whose value exceeds that of the right hand side can be reduced to the value of the latter, without cutting off any 0-1 point x satisfying (25). ‖

Note that if for some $i \in Q_M$ and $j \in N^-$ we have $i = i(j)$, then $(24)_{M,\varphi}$ has a term in $\bar{x}_i$ and one in x_i. Each such pair can obviously be reduced to a single term, with a corresponding adjustment of the constant on the right hand side. This in turn allows a further strengthening. We denote

$$Q_0^+ = \{i \in Q_M | i \ne i(j), \; \forall \; j \in N^-\}$$

$$Q_1^+ = \{i \in Q_M | i = i(j) \text{ for some } j \in N^-, \text{ and } \alpha_i^M > \beta_i\}$$

$$Q_0^- = \{i \in Q \backslash Q_M | i = i(j) \text{ for some } j \in N^-\}$$

$$Q_1^- = \{i \in Q_M | i = i(j) \text{ for some } j \in N^- \text{ and } \beta_i > \alpha_i^M\}$$

and let $Q^+ = Q_0^+ \cup Q_1^+$, $Q^- = Q_0^- \cup Q_1^-$.

Remark 9. The inequality $(24)_{M,\varphi}$ implies

$$(26)_{M,\varphi} \qquad \sum_{i \in Q^+} \gamma_i \bar{x}_i + \sum_{i \in Q^-} \gamma_i x_i \geq \gamma_0,$$

where

$$\gamma_0 = \max\{0, \alpha_0^M - \sum_{i \in Q_1^+} \beta_i - \sum_{i \in Q_1^-} \alpha_i^M\}$$

and

$$\gamma_i = \min\{\gamma_0, \gamma_i'\} \quad , \quad i \in Q^+ \cup Q^-,$$

with

$$\gamma_i' = \begin{cases} \alpha_i^M & i \in Q_0^+ \\[1em] \alpha_i^M - \beta_i & i \in Q_1^+ \\[1em] \beta_i & i \in Q_0^- \\[1em] \beta_i - \alpha_i^M & i \in Q_1^- \end{cases}$$

Proof. For $i \in Q_1^+$,

$$\alpha_i^M \bar{x}_i + \beta_i x_i = (\alpha_i^M - \beta_i)\bar{x}_i + \beta_i$$

while for $i \in Q_1^-$,

$$\alpha_i^M \bar{x}_i + \beta_i x_i = (\beta_i - \alpha_i^M)x_i + \alpha_i^M .$$

Substituting these expressions into $(24)_{M,\varphi}$ yields

$$\sum_{i \in Q^+} \gamma_i' \bar{x}_i + \sum_{i \in Q^-} \gamma_i' x_i \geq \gamma_0$$

and replacing γ_i' with $\min\{\gamma_0, \gamma_i'\}$ produces $(26)_{M,\varphi}$.

Theorem 12. For $M \subseteq N^+$, denote $C(M) = M \cup N^-$.

(i) If $C(M)$ is a minimal cover for (2), then $\alpha_i^M = \alpha_0^M$, $\forall i \in Q_M$, and $\beta_i = \alpha_0^M$, $\forall i \in Q_\varphi$ in $(24)_{M,\varphi}$.

(ii) If $C(M)$ is a minimal cover for (2) and $i(j) \in Q \backslash Q_M$, $\forall j \in N^-$ then $(26)_{M,\varphi}$ is the same as $(24)_{M,\varphi}$, and is of the form

$$(27)_{M,\varphi} \qquad \sum_{i \in Q^+} \bar{x}_i + \sum_{i \in Q^-} x_i \geq 1 .$$

(iii) If $C(M)$ is a minimal cover for (2) and $i(j) \in Q_M$ for some $j \in N^-$, then $(26)_{M,\varphi}$ is vacuous.

Proof. (i) Let $C(M)$ be a minimal cover for (2). Then

$$(28) \qquad \sum_{j \in C(M)} |a_j| \leq b - \sum_{j \in N^-} a_j + |a_k| \quad , \quad \forall k \in M \cup N^-$$

and therefore, for any $i \in Q_M$,

$$\overline{\sum_{j \in M | i \in Q_j}} a_j \geq \min_{k \in M} |a_k|$$

$$\geq \sum_{j \in M} a_j - b = \alpha_0^M \qquad \text{(from (28))},$$

which proves that $\alpha_i^M = \alpha_0^M$, $i \in Q_M$.

Also, for any $i \in Q_\varphi$ and $k \in N^-$ such that $i = i(k)$, from (28) we have

$$\overline{\sum_{j \in N^- | i = i(j)}} |a_j| \geq |a_k| \geq \sum_{i \in M} a_i - b = \alpha_0^M,$$

i.e., $\beta_i = \alpha_0^M$, $\forall j \in N^-$.

(ii) If $i(j) \in Q \backslash Q_M$, $\forall j \in N^-$, then $Q_i^+ \cup Q_i^- = \emptyset$, and $\gamma_i = \alpha_i^M$, $i \in Q^+$, $\gamma_i = \beta_i$, $i \in Q^-$. Thus, if this condition holds and $C(M)$ is a minimal cover for (2), then $(26)_{M,\varphi}$ is the same as $(24)_{M,\varphi}$, and it is of the form $(27)_{M,\varphi}$.

(iii) If $C(M)$ is a minimal cover for (2), then form (1), $(24)_{M,\varphi}$ is of the form

$$\sum_{i \in Q_M} \alpha_0^M \bar{x}_i + \sum_{i \in Q_\varphi} \alpha_0^M x_i \geq \alpha_0^M .$$

Now if $i(j_*) \in Q_M$ for some $j_* \in N^-$, then $\gamma_0 = 0$ and hence $\gamma_i = 0$, $i \in Q^+ \cup Q^-$. ‖

Note that $C(M)$ is a cover for (2) if and only if M is a cover for (2^+). If $C(M)$ is a minimal cover for (2), then M is a minimal cover for (2^+); but the converse is not true. On the other hand, if M is a minimal cover for (2^+) and $|a_j| \geq \min_{i \in M} a_i$ for all $j \in N^-$, then $C(M)$ is a minimal cover for (2).

Example 7. Consider the inequality

$$f(x) = 3x_1x_2x_3 + 3x_2x_4 + 4x_1x_4 - 2x_1x_5x_6 - x_3x_5 - 4x_1x_3x_6 \leq 2,$$

and let the sets Q_i, $i = 1,\ldots,6$, be numbered from left to right. Choosing $M = \{1,2,3\}$ and φ such that $i(4) = 1$, $i(5) = i(6) = 3$, one obtains the inequality (of the form $(24)_{M,\varphi}$)

$$7\bar{x}_1 + 6\bar{x}_2 + 3\bar{x}_3 + 7\bar{x}_4 + 2x_1 + 5x_3 \geq 8.$$

After reducing the terms involving the pairs $(\bar{x}_1, x_1)$ and $(\bar{x}_3, x_3)$, one obtains

$$5\bar{x}_1 + 6\bar{x}_2 + 7\bar{x}_4 + 2x_3 \geq 3$$

which in turn implies the inequality (of the form $(26)_{M,\varphi}$)

$$3\bar{x}_1 + 3\bar{x}_2 + 3\bar{x}_4 + 2x_3 \geq 3.$$

Now let $M = \{2\}$ and φ as above; then $C(M)$ is a minimal cover for $f(x) \leq 2$, and $i(j) \in Q \backslash M$, $\forall j \in N^-$. The corresponding inequality

$$\bar{x}_2 + \bar{x}_4 + x_1 + x_3 \geq 1$$

is of the form $(27)_{M,\varphi}$, since M and φ satisfy the condition of Theorem 12. ‖

Thus, when $M \subseteq N^+$ and $\varphi \in \Psi^-$ are chosen such that $C(M)$ is a minimal cover for (2) and $i(j) \in Q \backslash Q_M$ for all $j \in N^-$, $(26)_{M,\varphi}$ takes on the form $(27)_{M,\varphi}$ of a generalized set covering inequality. While the generalized set covering inequalities $(27)_{M,\varphi}$ all correspond to minimal covers $M \subseteq N^+$, the remaining generalized set covering inequalities implied by (2), corresponding to minimal covers $M \not\subseteq N^+$, can be derived by applying Theorem 11 to the family of inequalities $(22)_\varphi$, $\varphi \in \Psi^-$ (with M a minimal cover for $(22)_\varphi$), rather than directly to (2). The set of all generalized set covering inequalities corresponding to minimal covers for (2) has been shown by Granot and Hammer [16] to be equivalent to (in the sense of having the same 0-1 solutions as) the nonlinear inequality (2). Thus, whatever additional inequalities Theorem 11 produces, beyond the generalized set covering inequalities (27), come from covers other than minimal. In the context of linear inequalities, it is known [5,6] that canonical inequalities derived from minimal covers can usually be strengthened, and can never be weakened, by extending the covers. Unfortunately in the case of nonlinear inequalities, only the first part of this statement is true: extending a minimal cover may weaken the inequality associated with it.

Example 8. To show that extending a minimal cover can actually weaken the inequality derived from the cover, let

$$7x_2 x_5 x_6 + 6x_1 x_3 x_4 + 5x_2 x_4 + 2x_1 x_3 \leq 12,$$

with $Q_1 = \{2,5,6\}$, $Q_2 = \{1,3,4\}$, $Q_3 = \{2,4\}$ and $Q_4 = \{1,3\}$. Applying Theorem 11 and using the minimal cover $M = \{2,3,4\}$, we obtain the inequality

$$\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4 \geq 1$$

Extending now the minimal cover M to $\{1,2,3,4\}$, we obtain the inequality

$$8\bar{x}_1 + 8\bar{x}_2 + 8\bar{x}_3 + 8\bar{x}_4 + 7\bar{x}_5 + 7\bar{x}_6 \geq 8,$$

which is actually weaker than (c-dominated by) the first inequality.$\|$

Fortunately, the phenomenon illustrated by Example 8 can be precisely characterized.

Next we address the practically significant question as to when an inequality $(24)_{M,\varphi}$, where $M \in \mathcal{C}$, can be strengthened by expanding the set M.

As seen from the previous discussion, the presence of indices $i \in Q_M \cap Q_\varphi$ denotes a certain "weakness" of the inequality, since it offers a trivial way of strengthening it. We will therefore assume that, to start with, M is chosen such that $Q_M \cap Q_\varphi = \emptyset$, and that M is expanded into a set R such that $Q_R \cap Q_\varphi = \emptyset$ too.

<u>Theorem 13</u>. Let $M \in \mathcal{C}$, $S \subseteq N^+\backslash M$, $R = M \cup S$, and let M and S be such that $Q_R \cap Q_\varphi = \emptyset$. For $T \subseteq Q_M \cup Q_\varphi$, define

$$I(T) = (T \cap Q_M) \cup (Q_S \backslash Q_M)$$

and

$$\Delta(T) = \alpha_0^M - \sum_{i \in T \cap Q_M} \alpha_i^M - \sum_{i \in T \cap Q_\varphi} \beta_i .$$

Then the inequality

$(24)_R$ $\qquad \sum_{i \in Q_R} \alpha_i^R \bar{x}_i + \sum_{i \in Q_\varphi} \beta_i x_i \geq \alpha_0^R$

c-dominates $(24)_M$ if and only if

$$(29) \qquad \sum_{j \in S} (|I(T) \cap Q_j| - 1)a_j < \Delta(t)$$

for all $T \subseteq Q_M \cup Q_\varphi$ (including $T = \emptyset$) such that $\Delta(T) > 0$.

  <u>Proof</u>. From the definitions,

$$\alpha_0^R = \alpha_0^M + \sum_{j \in S} a_j$$

and for $i \in Q_S \backslash Q_M$,

$$(30) \qquad \alpha_i^R = \min\{\alpha_0^M + \sum_{j \in S} a_j, \overline{\sum_{j \in S | i \in Q_j}} a_j\}$$

$$= \overline{\sum_{j \in S | i \in Q_j}} a_j.$$

  To prove the "only if" part of the Theorem, suppose condition (29) is violated for $T^* \subseteq Q_M \cup Q_\varphi$ such that $\Delta(T^*) > 0$. Then let x^o be defined by

$$Q(x^o) = (Q_M \backslash T^*) \cup (Q_\varphi \cap T^*).$$

  If $T^* \neq \emptyset$, then $\Delta(T^*) > 0$ implies $\alpha_i^M < \alpha_0^M$, $\forall\ i \in T^* \cap Q_M$; hence for all $i \in T^* \cap Q_M$,

$$(31) \qquad \alpha_i^R = \alpha_i^M + \overline{\sum_{j \in S | i \in Q_j}} a_j .$$

  From the choice of $Q(x^o)$,

$$Q(\overline{x^o}) = I(T^*) \cup (Q_\varphi \backslash T^*) \cup [Q \backslash (Q_R \cup Q_\varphi)].$$

  Thus we have

$$\sum_{i \in Q_R} \alpha_i^R \bar{x}_i^o + \sum_{i \in Q_\varphi} \beta_i x_i^o = \sum_{i \in I(T^*)} \alpha_i^R + \sum_{i \in T^* \cap Q_\varphi} \beta_i$$

$$(32) \qquad = \sum_{i \in T^* \cap Q_M} \alpha_i^M + \sum_{i \in T^* \cap Q_\varphi} \beta_i + \sum_{i \in I(T^*)} \left(\sum_{j \in S \mid i \in Q_j} a_j \right)$$

$$= \sum_{i \in T^* \cap Q_M} \alpha_i^M + \sum_{i \in T^* \cap Q_\varphi} \beta_i + \sum_{j \in S} |I(T^*) \cap Q_j| a_j$$

$$(33) \qquad \geq \alpha_0^M + \sum_{j \in S} a_j = \alpha_0^R$$

where (32) follows from (30), (31), while the inequality (33) comes from the assumption that (29) is violated for T^*. Thus x^o satisfies $(24)_R$. However,

$$\sum_{i \in Q_M} \alpha_i^M \bar{x}_i^o + \sum_{i \in Q_\varphi} \beta_i x_i^o = \sum_{i \in T^* \cap Q_M} \alpha_i^M + \sum_{i \in T^* \cap Q_\varphi} \beta_i < a_0^M ,$$

since $\Delta(T^*) > 0$. Thus x^o violates $(24)_M$, hence $(24)_R$ does not c-dominate $(24)_M$.

To prove the "if" part of the Theorem, suppose (29) holds for all $T \subseteq Q_M \cup Q_\varphi$ such that $\Delta(T) > 0$. Then let $x^* \in \{0,1\}^q$ be any vector that violates $(24)_M$, and define

$$T^o = (Q_M \backslash Q(x^*)) \cup (Q_\varphi \cap Q(x^*)).$$

We have

$$\sum_{i \in T^o \cap Q_M} \alpha_i^M + \sum_{i \in T^o \cap Q_\varphi} \beta_i = \sum_{i \in Q_M} \alpha_i^M \bar{x}_i^* + \sum_{i \in Q_\varphi} \beta_i x_i^* < x_0^M ,$$

since x^* violates $(24)_M$. Thus $\Delta(T^o) > 0$. Further, if $T^o \neq \emptyset$, then $\alpha_i^M < x_0^M$, $\forall i \in T^o \cap Q_M$, and (31) holds for all $i \in T^o \cap Q_M$. Also, (30) holds for all $i \in Q_S \backslash Q_M$.

From the choice of T^o,

$$Q_R \cap Q(\bar{x}\star) \subseteq I(T^o)$$

and

$$Q_\varphi \cap Q(x\star) = T^o \cap Q_\varphi .$$

Therefore we have

$$\text{(34)} \quad \sum_{i \in Q_R} \alpha_i^R \bar{x}_i^\star + \sum_{i \in Q_\varphi} \beta_i x_i^\star \leq \sum_{i \in I(T^o)} \alpha_i^R + \sum_{i \in T^o \cap Q_\varphi} \beta_i$$

$$= \sum_{i \in T^o \cap Q_M} \alpha_i^M + \sum_{i \in T^o \cap Q_\varphi} \beta_i + \sum_{j \in S} |I(T^o) \cap Q_j| a_j$$

$$\text{(35)} \quad < \alpha_0^M + \sum_{j \in S} a_j = \alpha_0^R ,$$

where (34), just like (32), follows from (30), (31), and the strict inequality (35) follows since (29) is assumed to hold for T^o.

Thus $x\star$ violates $(24)_R$, i.e., $(24)_R$ c-dominates $(24)_M$.∥

At this point, it should be noted that when attempting to strengthen an inequality $(24)_M$ by expanding the cover M, it is not always possible to do so sequentially by introducing one new term at a time, as the following example illustrates.

Example 9. In the inequality

$$8x_1 x_2 + 5x_1 x_5 + 5x_1 x_6 + 4x_2 x_3 x_4 + x_3 x_5 + x_4 x_6 \leq 12,$$

let the sets Q_j, $j = 1,\ldots,6$, be indexed from left to right. Taking $M = \{1,2,3,4\}$, we obtain the inequality

$$10\bar{x}_1 + 10\bar{x}_2 + 4\bar{x}_3 + 4\bar{x}_4 + 5\bar{x}_5 + 5\bar{x}_6 \geq 10.$$

If we attempt to expand M by setting $R = M \cup \{5\}$, condition (29) is not met for $T = \{3,5\}$. Similarly, if we set $R = M \cup \{6\}$, (29) is violated for $T = \{4,6\}$. However, if we set $R = M \cup \{5,6\}$, then (29) is satisfied for all T such that $\Delta(T) > 0$. $\|$

We now focus on the case when $C(M)$ is a minimal cover for (2), i.e., when $\alpha_i^M = \beta_j = \alpha_0^M$ for all $i \in Q_M$ and $j \in Q_\varphi$.

<u>Corollary 13.1</u>. Let $M \subseteq N^+$ and $S \subseteq N^+ \backslash M$ be such that $Q_R \cap Q_\varphi = \emptyset$, where $R = M \cup S$, and let $C(M)$ be a minimal cover for (2). Then the inequality

$$(24)_R \qquad \sum_{i \in Q_R} \alpha_i^R \bar{x}_i + \sum_{i \in Q_\varphi} \beta_i x_i \geq \alpha_0^R$$

c-dominates $(24)_M$ if and only if

$$(36) \qquad \sum_{i \in Q_S \backslash Q_M} \alpha_i^R < \alpha_0^R .$$

Further, $(24)_R$ strictly c-dominates $(24)_M$ if and only if (36) holds and either

$$(37) \qquad \alpha_k^R < \alpha_0^R$$

for some $k \in Q_M$, or

$$(38) \qquad \beta_\ell < \alpha_0^R$$

for some $\ell \in Q_\varphi$.

<u>Proof</u>. If $C(M)$ is a minimal cover for (2), then since $\alpha_i^M = \beta_j = \alpha_0^M$, $\forall\ i \in Q_M$, $j \in Q_\varphi$, it follows that $\Delta(T) > 0$ implies $T = \emptyset$ and $\Delta(T) = \alpha_0^M$. Further, since $I(\emptyset) = Q_S \backslash Q_M$, condition (29) of Theorem 13 becomes

$$\sum_{j \in S} (|Q_S \backslash Q_M) \cap Q_j| - 1)a_j < \alpha_0^M$$

or

$$\sum_{i \in Q_S \setminus Q_M} \alpha_i^R = \sum_{i \in Q_S \setminus Q_M} (\sum_{j \in S | i \in Q_j} a_j) < \alpha_0^M + \sum_{j \in S} a_j = \alpha_0^R \, ,$$

which is (36). Thus (36) is necessary and sufficient for $(24)_R$ to c-dominate $(24)_M$.

Assume now that $(24)_R$ strictly c-dominates $(24)_M$. Then there exists $x^o \in \{0,1\}^q$ that satisfies $(24)_M$ (which implies $Q(x^o) \neq \emptyset$), but not $(24)_R$. Then there exists either $k \in Q(x^o) \cap Q_M$ such that (37) holds, or $\ell \in Q(x^o) \cap Q_\varphi$ such that (38) holds.

Conversely, if there exists $k \in Q_M$ such that $\alpha_k^R < \alpha_0^R$, then $x^* \in \{0,1\}^q$ defined by $Q(x^*) = \{k\}$ satisfies $(24)_M$ but not $(24)_R$; and an analogous argument holds if there exists $\ell \in Q_\varphi$ such that $\beta_\ell < \alpha_0^R$.‖

An important practical consequence of Corollary 13.1, which is used in the Algorithm of the next section, can be stated as follows. Like in section 2, we define for $M \subseteq N^+$

$$E_i(M) = \{j \in N^+ \mid |Q_j \setminus Q_M| = i\}, \quad i = 0,1,\ldots,p,$$

where $p = \max_{j \in N^+} |Q_j \setminus Q_M|$, and denote $E(M) = E_0(M) \cup E_1(M)$.

Corollary 13.2. Let M, S and R be as in Corollary 13.1. If $S \subseteq E(M)$, then $(24)_R$ c-dominates $(24)_M$.

Proof. Let $S \subseteq E(M)$ and denote $S_i = S \cap E_i(M)$, $i = 0,1$. Then $S = S_0 \cup S_1$, and

$$|(Q_S \setminus Q_M) \cap Q_j| = \begin{cases} 0 & \text{for } j \in S_0 \\ 1 & \text{for } j \in S_1 \end{cases}$$

Hence

(39) $$\sum_{i \in Q_S \setminus Q_M} \alpha_i^R = \sum_{j \in S_1} a_j < \alpha_0^M + \sum_{j \in S} a_j \, ,$$

i.e., (36) is satisfied.‖

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Thus any minimal cover $M \subseteq N^+$ for (2^+) can safely be extended to include all terms in $E(M)$, without weakening the inequality $(24)_M$. However, we can often go beyond $E(M)$, as will be clear when we restate Corollary 13.1 in slightly different form.

Corollary 13.3. Let M, S and R be as in Corollary 13.1, and let $R_i = R \cap E_i(M)$, $i = 1, \ldots, p$. Then the inequality $(24)_R$ c-dominates $(24)_M$ if and only if

$$(39) \qquad \sum_{i=2}^{p} [(i - 1) \sum_{j \in R_i} a_j] < \sum_{j \in R_0} a_j - b.$$

Further, $(24)_R$ strictly c-dominates $(24)_M$ if and only if (39) holds and either

$$(40) \qquad \overline{\sum_{j \in R | k \notin Q_j}} a_j > b$$

for some $k \in Q_M$, or

$$(41) \qquad \sum_{j \in R} a_j + \overline{\sum_{j \in N^- | \ell = i(j)}} a_j$$

for some $\ell \in Q_\varphi$.

Proof. We show that conditions (39), (40) and (41) are equivalent to conditions (36), (37) and (38) of Corollary 13.1.

For $i \in Q_S \setminus Q_M$,

$$\alpha_i^R = \overline{\sum_{j \in S | i \in Q_j}} a_j \quad (= \overline{\sum_{j \in R | i \in Q_j}} a_j, \text{ since for } j \in R \setminus S, \ i \notin Q_j),$$

while

$$\alpha_0^R = \sum_{j \in R} a_j - b.$$

Thus condition (36) amounts to

$$(42) \qquad \overline{\sum_{i \in Q_S \backslash Q_M}} \alpha^R_i = \sum_{j \in R} |(Q_S \backslash Q_M) \cap Q_j| a_j < \sum_{j \in R} a_j - b = \alpha^R_0 .$$

Now $R = \bigcup_{i=0}^{p} R_i$, and for $j \in R_i$, $|(Q_S \backslash Q_M) \cap Q_j| = i$, $i = 0,1,\ldots,p$. Hence from (42) we have

$$\sum_{j \in R_0} a_j - b > \sum_{i=2}^{p} (i \sum_{j \in R_i} a_j) - \sum_{i=2}^{p} (\sum_{j \in R_i} a_j)$$

$$= \sum_{i=2}^{p} [(i - 1) \sum_{j \in R_i} a_j] ,$$

which is precisely (39). Thus (39) is equivalent to (42), hence to (36), and this proves the first statement.

On the other hand, (37) can be restated as

$$\overline{\sum_{j \in R | k \in Q_j}} a_j < \sum_{j \in R} a_j - b$$

which is equivalent to (40). Also, (38) can be written as

$$\overline{\sum_{j \in N^- | \ell = i(j)}} |a_j| < \sum_{j \in R} a_j - b,$$

which is the same as (41). This proves the second statement.

Condition (39) of Corollary 13.3 gives the precise extent to which a minimal cover M for (2^+) (such that $Q_M \cap Q_\varphi = \emptyset$) can be extended beyond the sets $E_0(M)$ and $E_1(M)$, into sets $E_i(M)$ for $i \geq 2$. This is extensively used in the Algorithm described in section 4.

In particular, using Theorem 10 to replace the inequality (2) by the set of inequalities

$$(22)_\varphi \qquad f^+(x) + \sum_{j \in N^-} |a_j| \bar{x}_{i(j)} \leq \hat{b} \ (= b - \sum_{j \in N^-} a_j), \qquad \varphi \in \Psi^-,$$

where $i(j) = \varphi(Q_j)$, $j \in N^-$, we have

Corollary 13.4. Let M be a minimal cover for $(22)_\varphi$, let $S \subseteq N \backslash M$, $R = M \cup S$ and $R_i = R \cap E_i(M)$, $i = 0, 1, \ldots, p$. Assume M and S are such that $R \cap N^+ \cap Q_\varphi = \emptyset$. Then the inequality

$$(43)_R \qquad \sum_{i \in Q_{R \cap N^+}} \alpha_i^R \bar{x}_i + \sum_{i \in Q_{R \cap N^-}} \alpha_i^R x_i \geq \alpha_0^R$$

c-dominates the inequality

$$(43)_M \qquad \sum_{i \in Q_M \cap N^+} \bar{x}_i + \sum_{i \in Q_M \cap N^-} x_i \geq 1$$

if and only if

$$(44) \qquad \sum_{i=2}^{p} [(i-1) \sum_{j \in R_i} a_j] < \sum_{j \in R_0} a_j - \hat{b};$$

and $(43)_R$ strictly c-dominates $(43)_M$ if and only if (44) holds and there exists $k \in Q_M$ such that

$$(45) \qquad \sum_{j \in R | k \notin Q_j} a_j > \hat{b}.$$

Proof. By specializing Corollary 13.3 to inequality $(22)_\varphi$. ‖

The following example illustrates the usefulness of these results for obtaining a more compact equivalent linear system to a nonlinear inequality (2) than the set of generalized covering inequalities.

Example 10. Consider the inequality

(46) $10x_1x_9 + 9x_2x_8 + 8x_3x_7 + 8x_1x_6 + 8x_3x_4 + 5x_2x_5 \leq 20.$

There are 20 minimal covers M for (42), and the corresponding sets Q_M are shown in the table below.

Minimal Cover (M)	Q_M	Replaced by
# 1	1 2 3 7 8 9	B
2	1 2 6 8 9	D
3	1 2 3 4 8 9	B
4	1 2 5 8 9	D
5	1 3 6 7 9	E
6	1 3 4 7 9	E
7	1 2 3 5 7 9	B
8	1 3 4 6 9	E
9	1 2 5 6 9	D
10	1 2 3 4 5 9	B
11	1 2 3 6 7 8	B
12	2 3 4 7 8	C
13	2 3 5 7 8	C
14	1 2 3 4 6 8	B
15	1 2 5 6 8	D
16	2 3 4 5 8	C
17	1 3 4 6 7	E
18	1 2 3 5 6 7	B
19	2 3 4 5 7	C
20	1 2 3 4 5 6	B

Applying the Corollaries to minimal covers 20, 19, 15, and 17, respectively, we obtain the linear inequalities

(A) $\quad 18\bar{x}_1 + 14\bar{x}_2 + 16\bar{x}_3 + 8\bar{x}_4 + 5\bar{x}_5 + 8\bar{x}_6 + 8\bar{x}_7 + 9\bar{x}_8 + 10\bar{x}_9 \geq 28$

(B) $\quad\quad\quad\quad 10\bar{x}_2 + 10\bar{x}_3 + 8\bar{x}_4 + 5\bar{x}_5 \quad\quad + 8\bar{x}_7 + 9\bar{x}_8 \quad\quad\quad \geq 10$

(C) $\quad 12\bar{x}_1 + 12\bar{x}_2 \quad\quad\quad\quad + 5\bar{x}_5 + 8\bar{x}_6 \quad\quad + 9\bar{x}_8 + 10\bar{x}_9 \geq 12$

(D) $\quad 14\bar{x}_1 \quad\quad + 14\bar{x}_3 + 8\bar{x}_4 \quad\quad + 8\bar{x}_6 + 8\bar{x}_7 \quad\quad + 10\bar{x}_9 \geq 14,$

which c-dominate all the set covering inequalities corresponding to the minimal covers. Whereas a system of 20 set covering inequalities were previously required to linearize (46), the system of four inequalities (A)-(D) is equivalent to (46).‖

We conclude this section by considering whether the multilinear inequality (2) can best be linearized by applying the relevant results directly to inequality (2) in the manner previously described, or alternatively applying them to an equivalent set of inequalities implied by (2). In particular, if we consider (2) to be a linear inequality in the 0-1 variables

$$y_j = \prod_{i \in Q_j} x_i , \quad\quad\quad j \in N$$

and denote it by $(2)_y$, then w.l.o.g. we may assume that $a_j > 0$ for all $j \in N$, and that $a_1 \geq a_2 \geq \ldots \geq a_n$. Then applying the results of [6], [5], we can replace $(2)_y$ by the equivalent set of canonical inequalities

$$(47)_S \quad\quad \sum_{j \in \mathcal{E}(S)} y_j \leq |S| - 1 , \quad S \in \mathcal{K} .$$

Here $\mathcal{E}(S)$ is the **extension** of S, defined as

$$\mathcal{E}(S) = S \cup \{ j \in N \backslash S \mid j < j_1 \},$$

with j_1 = $\min_S$ j; while $\mathcal{K}$ is the family of __strong__ covers for $(2)_y$, where
a minimal cover S is called strong if there exists no minimal cover $T \neq S$
such that $|T| = |S|$ and $\mathcal{C}(S) \subseteq \mathcal{C}(T)$.

For any given $S \in \mathcal{K}$, rewriting $(47)_S$ in terms of x, we can linearize it
using the above results. If we do this for every $S \in \mathcal{K}$, we obtain a new
linearization of (2), different from the one discussed earlier. Naturally,
the question arises as to how this new linearization compares with the one
discussed above. Both approaches were implemented and tested, and the computa-
tional results are reported in section 5.

In [5], a procedure was given for strengthening canonical inequalities
of the form $(47)_S$ by increasing their left hand side coefficients that satisfy
a certain condition. Though the strengthened inequality strictly dominates and
sometimes strictly c-dominates the canonical inequality $(47)_S$, it is not
necessarily true that linearizing the strengthened inequality is preferable
to linearizing the inequality $(47)_S$. We have derived some sufficient condi-
tions for the linear inequality associated with a certain cover M for the
strengthened inequality, to dominate the linear inequality associated with
the same cover M for the canonical inequality $(47)_S$. However, in our computa-
tional experiments we found that the particular sufficient condition that
we have implemented was rarely satisfied; therefore we have not pursued any
further the idea of using strengthened inequalities instead of the family (47).

4. An Algorithm for Solving Multilinear 0-1 Programs

Next we address the multilinear 0-1 program

$$\text{Max} \sum_{i \in Q} c_i x_i$$

(MLP)
$$\sum_{j \in N_k} a_{kj} \left(\overline{\sum_{i \in Q_{kj}}} x_i \right) \leq b_k \quad , \quad k \in K$$

$$x_i = 0 \text{ or } 1, \quad i \in Q$$

where

$$Q = \bigcup_{\substack{k \in K \\ j \in N_k}} Q_{kj} \; .$$

The algorithm that we present below, like the one by Granot, Granot and Kallberg [15], generates some linear inequalities implied by the constraint set of (MLP), and solves the resulting linear 0-1 program, which is a relaxation of (MLP). At iteration t, let this linear 0-1 program be denoted (P_t). If an optimal solution to (P_t) is feasible for (MLP), then it is optimal for (MLP) and we stop. Otherwise we generate a new set of linear inequalities implied by the constraints of (MLP), such that the new inequalities cut off the solution to (P_t), and solve the linear 0-1 program (P_{t+1}) obtained from (P_t) by adding the new inequalities. Since at every iteration the solution to the current problem (P_t) is cut off, the algorithm is obviously finite.

Our procedure differs from that of [15, 14] mainly in that we use a more compact linearization, based on the theory of sections 2-3. To be more specific, we start with a set covering inequality associated with a minimal cover, but then use Theorem 13 and its Corollaries to extend the cover so as to obtain as strong an inequality as the conditions of the Corollary permit. Experience shows that the proportion of minimal covers that be extended is very high (90% is a typical case) and tends to increase with the number of terms per constraint. Since the use of extended covers tends to produce smaller cardinality linear equivalents of each nonlinear inequality, it can also be expected to reduce the number of iterations needed to solve (MLP). This is indeed the case, except for problems with very few terms per constraint, as shown by the computational experience discussed in the next section.

While the procedure outlined above is finite, it may take many iterations. We found it therefore preferable not to solve (P_t) exactly at every iteration, but use a heuristic to find an approximate solution. We proceed this way until, at some iteration t, an approximate solution to (P_t) is found to be feasible to [MLP]. At that point we replace the heuristic by an exact algorithm. The particular heuristic that we use on the sequence of linear 0-1 programs (P_t) is the Pivot and Complement procedure of Balas and Martin [7]. When we switch to an exact algorithm, we use a branch and bound/implicit enumeration procedure implemented by C. H. Martin.

Another deviation from the above outline is that we found it convenient to periodically remove some of the linear inequalities generated earlier. This is done according to a particular procedure so as to insure that convergence is maintained.

Finally, to facilitate the search for minimal covers and their extensions, used in the linearization procedure, we start the algorithm by ordering once and for all the terms of each constraint according to decreasing absolute values of their coefficients.

As a starting solution we use the optimal solution to the unconstrained problem, i.e., x^0 defined by $x_i^0 = 1$ if $c_i > 0$ and $x_i^0 = 0$ otherwise.

A flowchart of the algorithm is shown in fig. 1.

The heart of our procedure is of course the generation of linear inequalities. Since the conditions of Theorem 13 and its corollaries do not uniquely determine the "best" inequality that meets them, we have to describe the particular algorithm that we use to generate the inequalities.

First, it should be stated that at every iteration we generate one linear inequality from every inequality of (MLP) violated by the current solution x^0, except for the first iteration, when we generate one linear
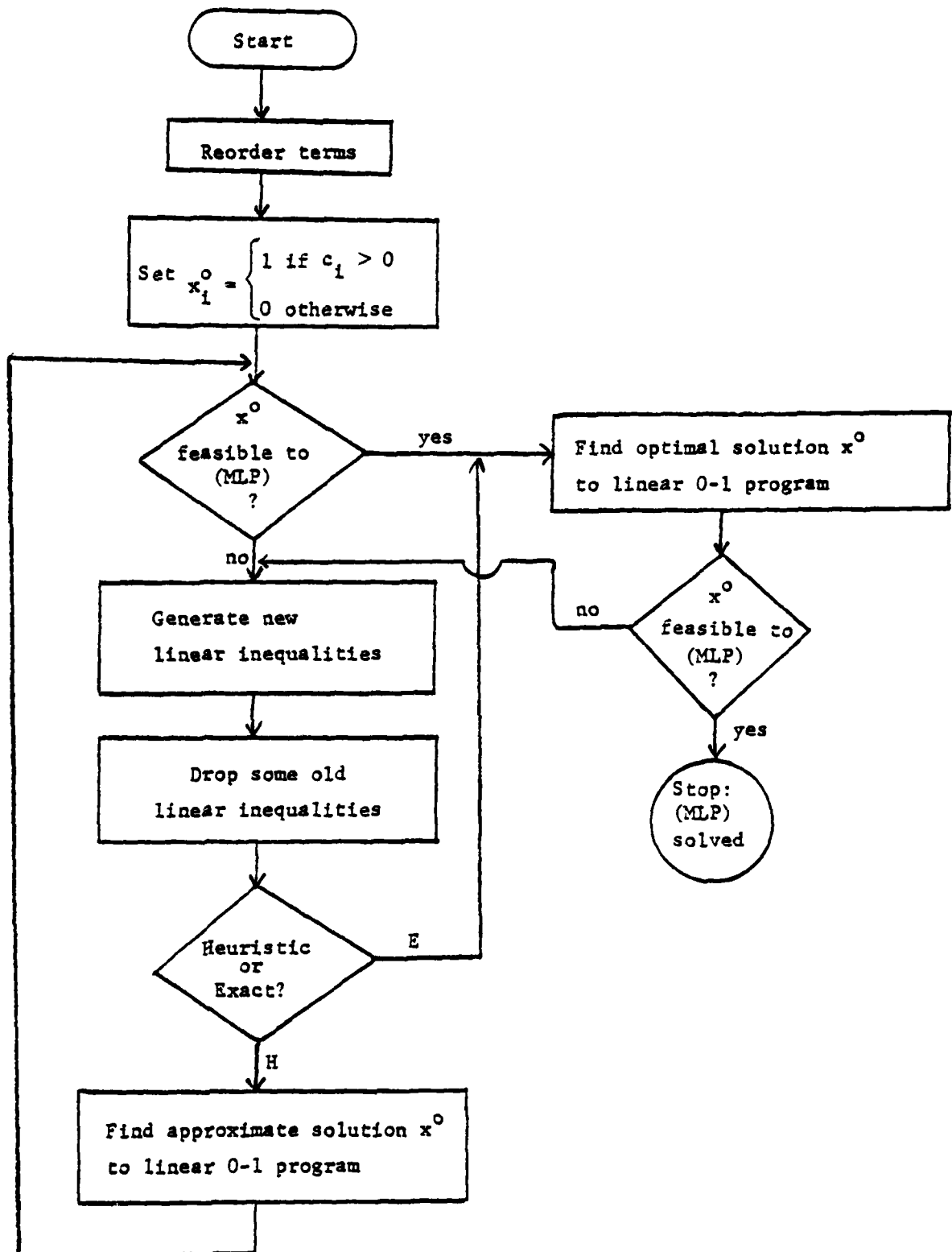
Fig. 1. Flowchart of the algorithm

inequality (using the cover $M = N$) from every constraint of (MLP), whether violated or not (the exception was adopted as a result of computational experimenting). To describe the procedure, let

$$(2) \qquad \sum_{j \in N} a_j (\prod_{i \in Q_j} x_i) \leq b$$

be one of the inequalities violated by x^o, and let $|a_1| \geq |a_2| > \dots \geq |a_n|$.

Denote

$$P^+(x^o) = \{ j \in N^+ | \prod_{i \in Q_j} x_i^o = 1 \},$$

$$P^-(x^o) = \{ j \in N^- | \prod_{i \in Q_j} x_i^o = 0 \},$$

with

$$P(x^o) = P^+(x^o) \cup P^-(x^o) ,$$

and define

$$\Phi^-(x^o) = \{ \varphi \in \Phi^- | x_{i(j)}^o = 0, \ \forall \ j \in P^-(x^o) \}.$$

If x^o violates (2), it also violates the inequality

$$(22)_\varphi \qquad \sum_{j \in N^+} a_j (\prod_{i \in Q_j} x_i) + \sum_{j \in N^-} |a_j| \bar{x}_{i(j)} \leq \hat{b} \ (= b - \sum_{j \in N^-} a_j)$$

for every $\varphi \in \Phi^-(x^o)$, and $\Phi^-(x^o) \neq \emptyset$ by definition. Thus, given x^o and the family of inequalities $(22)_\varphi$, $\varphi \in \Phi(x^o)$ (corresponding to a particular inequality (2)), violated by x^o, our cut generating algorithm consists of the following sequence of steps:

1. Finding a convenient minimal cover $M \subseteq N$, i.e., such that x^o violates the generalized covering inequality corresponding to M.

2. Extending M to a maximal set $R \subseteq N^+$ satisfying the condition (44).

3. Choosing a convenient mapping $\varphi \in \hat{\Phi}^-(x^0)$, i.e., one that avoids as much as possible producing nonzero coefficients for complementary pairs of variables, and including $N \setminus M$ in R.

A discussion of each step follows.

1. Let $P(x^0) = \{i_1, \ldots, i_t\}$ be ordered by the same rule as N, i.e., $i_k < i_{k+1}$, $k = 1, \ldots, t - 1$. Let $j \in \{1, \ldots, t\}$ be the largest integer such that $\{i_j, i_{j+1}, \ldots, i_t\}$ is a cover, and let $\ell \in \{j, j + 1, \ldots, t\}$ be the smallest integer such that

$$M = \{i_j, i_{j+1}, \ldots, i_\ell\}$$

is a cover for (2). Then, obviously, M is minimal. Also, M is a minimal cover for $(22)_\varphi$ for any $\varphi \in \hat{\Phi}^-(x^0)$. Further, for any $\varphi \in \hat{\Phi}^-(x^0)$, M satisfies the requirement of Corollary 13.4, i.e., $M \cap N^+ \cap W_\varphi = \emptyset$; and for any such φ, the generalized covering inequality

$$\sum_{i \in Q_{M \cap N^+}} \bar{x}_i + \sum_{i \in Q_{M \cap N^-}} x_i \geq 1$$

corresponding to M is violated by x^0.

2. Construct the extension R of M as follows. Define

$$E_i(M) = \{j \in N \mid |Q_j \setminus Q_M| = i\}, \quad i = 0, 1, \ldots, p,$$

and set $E(M) = E_0(M) \cup E_1(M)$. Note that $N^- \subseteq E(M)$. First include in R the set $E(M) \cap N^+$. Next, for $i = 2, \ldots, p$, consider the elements of $E_i(M)$ (which all belong to N^+) in order of increasing a_j, and include into R as many as can be included without violating condition (44). If all $j \in E_i(M)$ can be included, set $i = i + 1$ and repeat. Otherwise stop with the last element of $E_i(M)$ whose inclusion into R does not lead to a violation of (44).

3. To choose the mapping $\varphi \in \bar{\vartheta}^-(x^0)$, for $j \in N^- \cap M$, let $\varphi(Q_j)$ be the first index $i \in Q_j$ such that $x_i^0 = 0$. For $j \in N^- \backslash M$, let R be the extended set resulting at the end of step 2 and let α_i^R, $i \in Q \cup \{0\}$, be the corresponding coefficient values, where R and the α_i^R are updated by combining variables and their complements whenever such pairs occur. Since it is possible for either $\bar{x}_i$ or x_i (but never for both) to appear in the resulting inequality $(24)_{R,\varphi}$, we partition Q into Q_R^+, Q_R^- and Q^0, where

$$Q_R^+ = \{i \in Q_R \mid \bar{x}_i \text{ appears in } (24)_{R,\varphi}\} \ ,$$

$$Q_R^- = \{i \in Q_R \mid x_i \text{ appears in } (24)_{R,\varphi}\} \ ,$$

and

$$Q^0 = Q \backslash (Q_R^+ \cup Q_R^-).$$

We then choose $i(j) = \varphi(Q_j)$ according to the following rule:

If $Q_j \backslash Q_R \neq \emptyset$, let $i(j)$ be the first index in $Q_j \backslash Q_R$.

If $Q_j \backslash Q_R = \emptyset$ but $Q_j^* \equiv Q_j \cap Q_R^- \neq \emptyset$, let $i(j) = h$ where $\alpha_h^R = \max\limits_{i \in Q_j^*} \alpha_i^R$.

Otherwise, let $i(j) = k$, where $\alpha_k^R = \min\limits_{i \in Q_j} \alpha_i^R$.

Once $i(j)$ is selected, set $R \leftarrow R \cup \{i(j)\}$, update $\alpha_{i(j)}^R$ and α_0^R, as well as Q_R^+, Q_R^- and Q^0 (combining variables, if necessary), and proceed to the next $j \in N^- \backslash M$.

Having generated the linear inequality, we eliminate the complemented variables, i.e., restate the inequality in the original variables, and add it to the current linear 0-1 program.

Next we illustrate the procedure on an example.

Example 11. Consider the multilinear inequality

(48) $16x_2x_4x_5 - 10x_2x_6 + 10x_1x_2x_3 + 5x_1x_5 - 4x_5x_7 + 4x_3x_5 \leq 1,$

which is violated by $x^o = (1,1,1,0,1,1,0)$. We have $Q_1 = \{2,4,5\}$, $Q_2 = \{2,6\}$, $Q_3 = \{1,2,3\}$, $Q_4 = \{1,5\}$, $Q_5 = \{5,7\}$, $Q_6 = \{3,5\}$. Further, $N^+ = \{1,3,4,6\}$, $N^- = \{2,5\}$, and

$$P^+(x^o) = \{3,4,6\}, \quad P^-(x^o) = \{5\} \quad P(x^o) = \{3,4,5,6\}.$$

The corresponding inequality with positive coefficients is (in general form)

(49) $\quad 16x_2x_4x_5 + 10\bar{x}_{i(2)} + 10x_1x_2x_3 + 5x_1x_5 + 4\bar{x}_{i(5)} + 4x_3x_5 \leq 15,$

where $i(2)$ and $i(5)$ depend on the choice of $\varphi \in \bar{\Phi}^-(x^o)$. Since in this case $\bar{\Phi}^-(x^o)$ is the set of those φ such that $\varphi(Q_5) = i(5) = 7$, only the choice of $i(2)$ remains open.

1. We identify the minimal cover $M = \{3,4,5\}$ for (48), which is also a minimal cover for (49). We have

$$Q_{M\cap N^+} = \{1,2,3,5\}, \qquad Q_{M\cap N^-} = \{7\},$$

and applying Theorem 11 to (49), we obtain for the minimal cover M the generalized covering inequality

(50) $\quad \bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_5 + x_7 \geq 1$

violated by x^o.

2. We identify the sets

$$E_0(M) = \{3,4,5,6\} , \quad E_1(M) = \{1,2\},$$

and since $E_i(M) = \emptyset$ for $i \geq 2$, we have $R = E(M) \cap N^+ = \{1,3,4,6\}$.

3. For $j = 5$ ($j \in N^- \cap M$), we set $i(5) = 7$, since $x_7^o = 0$, and update R by including $\{5\}$. For $j = 2$ ($j \in N^- \backslash M$), we set $i(2) = 6$, since $Q_2 \backslash Q_R = \{6\}$,

and update R by including $\{2\}$. Thus $R = \{1,2,3,4,5,6\}$, and by applying
Theorem 11 to (49) (with $i(2) = 6$ and $i(5) = 7$), we obtain for the extended
cover R the inequality

(51) $\qquad 15\bar{x}_1 + 26\bar{x}_2 + 14\bar{x}_3 + 16\bar{x}_4 + 25\bar{x}_5 + 10x_6 + 4x_7 > 34$

which is also violated by x^o, and which strictly c-dominates the generalized
covering inequality (50). ‖

As mentioned earlier, we found it necessary to periodically remove
inequalities from the linear 0-1 program in order to keep its size within
manageable limits. The cut dropping procedure operates as follows. The set V
of all inequalities generated during the procedure is partitioned into
3 subsets. V_1 contains exactly one inequality generated at each iteration,
namely the one derived from the most violated constraint of (MLP). Cuts in
V_1 are never removed, as a guarantee that every solution to the linear 0-1
program generated during the procedure is cut off by at least one inequality.
V_2 consists of all inequalities associated with extended covers and not
contained in V_1, whereas V_3 consists of the remaining inequalities (i.e.,
those associated with minimal covers that could not be extended).

Wh..never the number of inequalities in the linear 0-1 program attains
a predetermined threshold value Δ, all inequalities in V_3 not binding at the
current solution are dropped. The subset V_3 is our first preference for
dropping, since it usually consists of the weakest inequalities of the current
system. If removing the nonbinding inequalities in V_3 is not sufficient for
accommodating all the inequalities generated at the current iteration, then
the nonbinding inequalities in V_2 are also dropped. Finally, if removing all

the nonbinding inequalities of V_3 and V_2 is still insufficient, we drop an appropriate number of binding inequalities in V_3 and, if necessary, in V_2.

This completes the description of the main version of our algorithm, henceforth called Algorithm I. Two additional versions of the algorithm were implemented, which will now be briefly described.

Algorithm II differs from Algorithm I in that it generates linear inequalities not directly from an inequality (2) of (MLP), but from an extended canonical inequality implied by $(2)_y$, as described at the end of section 3. The choice of the inequality (2), respectively $(2)_y$, as well as that of the minimal cover M, is the same as in Algorithm I. Another minimal cover C is then identified, such that $|C| = |M|$ and $\mathcal{C}(M) \subseteq \mathcal{C}(C)$ (preferably, but not necessarily, $C \neq M$). The cut generating procedure described above is then applied to the canonical inequality defined by $\mathcal{C}(M)$ and expressed in terms of x, for which M is still a minimal cover. Everything else is as in Algorithm I.

Finally, Algorithm III differs from the other two versions by the fact that it derives only generalized covering inequalities corresponding to minimal covers, without attempting to strengthen them by extending the covers. For this version, the choice of the minimal cover is done differently, namely by setting $M = \{i_1, \ldots, i_k\}$, where k is the smallest integer such that M is a cover. As a result, M (which is of course minimal) is of smaller cardinality than the cover selected in Algorithm 1 which in the absence of the extension procedure is preferable. The superiority of this choice of minimal cover for this particular algorithm was unequivocally supported in the computational testing. The other ingredients of Algorithm III are the same as those of I and II. Algorithm III should be viewed as our version of the Granot and Granot algorithm [14]; the differences from the latter (improvements in our view) having been adopted in order to make it comparable with Algorithms I and II.

Algorithm 1, which is by far the most efficient of the three procedures implemented, was also run in the heuristic mode, i.e., by removing all steps subsequent to the finding of a feasible solution to (MLP). The purpose of this exercise was to obtain information on the quality of the solutions obtainable by such an approach.

5. Computational Results

The algorithms discussed above were coded in FORTRAN and tested on a DEC 20/60 on a series of randomly generated test problems.

The first set of test problems consists of 60 multilinear 0-1 programs, 5 in each of 12 classes that differ among themselves in the number of terms per constraint. The number of constraints and variables (denoted by m and n respectively) is the same in all of these problems (m = 10, n = 30), and the number of terms per constraint is randomly drawn from a uniform distribution on the interval $[3, T]$, where T is shown in Table 1. The constraint coefficients a_{kj} are integers uniformly distributed on $[-5, 15]$, while the b_k are integers drawn from a uniform distribution on $(0.3s_k, 0.8s_k)$, where $s_k = \Sigma_j a_{kj}$. The cost coefficients c_j are uniformly distributed integers on $[1, 20]$. Finally, the number of variables per term is uniformly distributed on $[2,6]$. The results are shown in Table 1.

All test problems were run under two kinds of limitations, shown in the tables: a time limit (3, 5 or 10 minutes, depending on the phenomenon studied) and a limit of 150 on the number of iterations, hence on the number of nonremovable inequalities generated, due to space limitations. The latter limit is different from the threshold value Δ that triggers the cut dropping routine. In Algorithms I and II, after some experimentation Δ was set to 2n, i.e., twice the number of variables; whereas in Algorithm III computational tests indicated a higher value, and $\Delta = 150$ was adopted.

All CPU times reported are exclusive of input/output time and the preprocessing time required to sort the (absolute values of the) coefficients of the multilinear constraints. The maximum input time for any of the test problems was 0.416 seconds, and the maximum preprocessing time was 0.093 seconds.

Table 1. Number of problems solved and average CPU time (seconds).[1,2,3]

m	n	T	Algorithm I		Algorithm II		Algorithm III	
			No. Solved	Time	No. Solved	Time	No. Solved	Time
10	30	10	5	38.7	5	16.8	5	17.9
10	30	20	5	4.2	5	4.2	5	2.6
10	30	30	5	91.8	4	77.0	3	122.3
10	30	40	5	57.8	3	81.7	3	85.2
10	30	50	5	95.2	2	110.7	2	112.1
10	30	60	4	56.5	2	109.9	1	165.8

1. 5 problems per class.

2. Time averaged for all 5 problems

3. Limit set to 3 minutes CPU time or 150 iterations per problem.

Table 1 shows that although Algorithm III performs somewhat better than Algorithm I on the problems with T = 10 and T = 20 (i.e., with 5 and 10 constraints on the average, respectively), its performance quickly deteriorates for higher values of T, as reflected in the sharply decreasing number of problems solved within the limits allowed. At the same time, the performance of Algorithm 1 is only moderately affected by the increase of T. As to

Algorithm II, its performance is not better than that of III on the problems
with small T, and considerably worse than that of Algorithm I on the problems
with large T. Thus the performance of Algorithm II will not be further pursued.

Table 2 compares the performance of Algorithms I and III on the same
set of problems with a time limit of 10 instead of 3 minutes.

Table 2. Number of problems solved and average CPU time (seconds).[1,2,3]

			Algorithm I		Algorithm III	
m	n	T	No. Solved	Time	No. Solved	Time
10	30	10	5	38.7	5	17.9
10	30	20	5	4.2	5	2.6
10	30	30	5	91.8	5	257.5
10	30	40	5	57.8	3	253.2
10	30	50	5	95.2	2	333.4
10	30	60	5	81.7	2	344.9

1. 5 problems per class.

2. Time averaged for all 5 problems.

3. Limit set to 10 minutes CPU time or 150 iterations per problem.

The results show an even sharper contrast between the sensitivity of
the two algorithms to an increase in the number of terms per constraints.
We conclude that the more compact linearization based on the theory of
sections 2-3 definitely pays off for problems with more than 20 terms per
constraint.

In Table 3 we compare the average number of iterations and cuts (linear inequalities) generated, in order to better understand the difference in the performance of the two algorithms. We see that as T is increased from, say, 30 to 60, the number of iterations and cuts increases by more than 300% for Algorithm III, as opposed to 10-15% for Algorithm I. On the other hand, while the percentage of covers that can be extended (in Algorithm I)

Table 3. Number of iterations and of cuts.[1,2,3]

m	n	T	Algorithm I			Algorithm III	
			Iterations	Cuts	Percent Covers Extended	Iterations	Cuts
10	30	10	6.6	32.6	89.0	9.0	32.4
10	30	20	4.2	25.2	94.6	6.8	27.4
10	30	30	8.8	41.0	94.2	26.0	108.0
10	30	40	9.2	38.6	95.1	57.2	212.0
10	30	50	9.6	41.8	98.9	78.2	340.6
10	30	60	10.2	45.6	98.8	100.0	455.2

1. 5 problems per class.

2. Values averaged for all 5 problems.

3. Limit set to 10 minutes CPU time or 150 iterations per problem.

increases with T, the increase is only modest, since this percentage is high to begin with (i.e., for all problem classes). This modest increase cannot fully account for the sharply increasing difference in the number of iterations required by the two Algorithms. What the table does not show, however, is that as the number of terms per constraint increases, not only does the percentage of covers that can be extended increase, but more importantly, there

is a significant increase in the _extent_ to which every minimal cover can
be extended: with more terms per constraint, many more indices are included
in the extension of each cover.

In Tables 4 and 5 we illustrate the effect of an increase in the number
of variables and constraints, respectively, on the performance of Algorithm I.

Table 4. Effect of an increase in the number of variables (Algorithm I)[1,2,3]

m	n	T	No. Solved	Time (seconds)	Iterations	Cuts	Percent Covers Extended
10	30	30	5	91.8	8.8	41.0	94.2
10	40	30	4	104.6	11.6	43.6	96.6
10	50	30	4	82.5	9.2	40.2	95.1

1. 5 problems per class.

2. Values averaged for all 5 problems.

3. Limit set to 3 minutes CPU time or 150 iterations per problem.

Table 5. Effect of an increase in the number of constraints (Algorithm I)[1,2,3]

m	n	T	No. Solved	Time (seconds)	Iterations	Cuts	Percent Covers Extended
5	30	30	5	5.5	6.0	17.4	95.2
10	30	30	5	91.8	8.8	41.0	94.2
15	30	30	3	183.9	12.6	68.2	96.0
20	30	30	2	253.8	29.2	144.6	93.1

1. 5 problems per class.

2. Values averaged for all 5 problems

3. Limit set to 5 minutes CPU time or 150 iterations per problem.

Table 4 shows that as the number of variables increases from 30 to 40 to 50, the number of problems that the algorithm is able to solve within 3 minutes and 150 iterations drops from 5 to 4. This is of course to be expected, since the number of variables increases to the same extent in the linear 0-1 program as in (MLP). Table 5 shows a steady deterioriation of performance as the number of constraints increases. This is due to the fact that the number of inequalities in the linear equivalent of (MLP) sharply rises with the number of constraints of (MLP), hence so does the number of iterations required to generate a relevant subset of the linear inequalities.

In the last two tables we finally examine the performance of Algorithm I in the heuristic mode. When used as a heuristic, Algorithm I stops at the first (approximate) solution of the linear 0-1 program (found by Pivot and Complement) that is feasible to (MLP).

Table 6. Algorithm I in the heuristic mode[1,2,3]

m	n	T	No. solved	Iterations	Time (seconds)	Proximity to LP bound	Proximity to Integer Optimum
10	30	20	5	4.2	2.0	1.7%	0.12%
10	30	30	5	7.4	7.6	2.9%	0.16%
10	30	40	5	6.8	4.6	2.4%	0.07%
10	30	50	5	8.6	10.1	2.2%	0.00%
10	30	60	5	10.0	15.4	2.7%	0.22%

1. 5 problems per class

2. Values averaged for all 5 problems.

3. Limit set to 3 minutes CPU time per problem.

The linear programming solution to the last linear 0-1 program (more precisely, the lowest value of any LP solved during the procedure), rounded down to the nearest integer, provides an upper bound for the optimum of (MLP), which we call the LP bound. This bound is guaranteed, but in most cases not tight. For the problems of Table 6 the integer optimum is also known, so the quality of the heuristic solution can be measured against the actual optimum. For the problems of Table 7 this is not the case, and the only measure available is the LP bound. On both counts, the quality of the solutions obtained by using Algorithm I in the heuristic mode seems excellent, and the computational effort is modest.

Table 7. Additional tests with the heuristic[1,2,3,]

m	n	T	No. Solved	Iterations	Time (seconds)	Proximity to LP optimum
10	30	70	4	29.0	64.1	2.4%[4]
10	40	30	5	2.1	16.9	2.9%
10	50	30	5	8.8	19.5	1.8%
10	50	40	5	10.2	26.5	1.8%
10	50	50	5	11.4	29.1	1.5%
5	100	30	5	8.2	14.8	0.7%
5	100	50	5	15.4	65.4	0.8%
5	150	30	5	6.8	16.9	0.4%

1. 5 problems per class.

2. Values averaged for all 5 problems.

3. Limit set to 3 minutes CPU time per problem.

4. Average for the 4 solved problems.

We conclude from this computational study that Algorithm I, based on the linearization of sections 2-3 is an efficient procedure for solving multilinear 0-1 programs to optimality. In particular, problems having more than 20 terms per constraint have now been opened up to exact solution. The use of the first phase of the algorithm as a heuristic is also an attractive option for problems with many constraints and/or variables, in that high quality solutions can be obtained at a modest computational cost.

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Any real-valued nonlinear function in 0-1 variables can be rewritten as a multilinear function. We discuss classes of lower and upper bounding linear expressions for multilinear functions in 0-1 variables. For any multilinear inequality in 0-1 variables, we define an equivalent family of linear inequalities. This family contains the set of generalized covering inequalities defined by Granot and Hammer. Several results concerning the relative strengths of inequalities within this family are presented. An algorithm for the general
(Over)

multilinear 0-1 program is given, and computational experience with the algorithm applied to randomly generated problems is discussed. The use of the general procedure as an effective heuristic for multilinear 0-1 programs is also demonstrated.